

Solution of Regular Problems:

$$1. \quad (a) \quad Y = \frac{1}{n} \sum_{i=1}^n X_i \quad E\{X_i\} = 0.5 \quad \text{Var}(X_i) = \frac{(0.5)^2}{3} = \frac{1}{12}$$

$$E\{Y\} = \frac{1}{n} E\left\{\sum_{i=1}^n X_i\right\} = \frac{1}{n} \sum_{i=1}^n m_X = m_X = 0.5$$

$$\sigma_Y^2 = \text{Var}(Y) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1}{n^2} n \sigma_X^2 = \frac{\sigma_X^2}{n} = \frac{1}{12n} = \frac{1}{12 \cdot 64} = 0.001302$$

$$(b) \quad f_Y(v) = \frac{e^{-\frac{(v-0.5)^2}{\frac{1}{12n}}}}{\sqrt{\frac{2\pi}{n \cdot 12}}} = \frac{e^{-\frac{(v-0.5)^2}{2 \cdot (0.001302)}}}{\sqrt{2\pi \cdot 0.036}}$$

$$(c) \quad 15\% \text{ of } 0.5 \text{ is } 0.075$$

$$P\{|Y - 0.5| < 0.075\} = P\{|Z| < \frac{0.075}{0.036}\} \\ = P\{|Z| < 2.08\} = \Phi(2.08) - \Phi(-2.08) = 0.981167 - 0.018833 = 0.962334$$

$$\text{where } Z = \frac{Y - 0.5}{0.036} = \text{unit Gaussian variable}$$

$$(d) \quad \text{margin of error} = 5\% \text{ of mean} = 0.5 \cdot 0.5 = 0.025$$

$$\text{We want same probability: } P\{|Y - 0.5| < 0.025\} = 0.962334$$

$$\text{Hence we have for } Z = \frac{Y - 0.5}{\sigma_Y} \text{ and } P\{|Z| < Z_c\} = 0.962334 \text{ and}$$

$$Z_c = \frac{0.025}{\sigma_Y} = 2.08 \text{ from the inverse Gaussian Table}$$

$$\sigma_Y = \frac{1}{\sqrt{12n}}$$

$$0.025 \sqrt{12n} = 2.08$$

$$n = \frac{1}{12} \left(\frac{2.08}{0.025} \right)^2 = 577 \quad \text{samples}$$

$$2. \quad m_X = 2, \quad m_Y = 5, \quad \sigma_X = 3, \quad \sigma_Y = 2, \quad C_{XY} = 5$$

$$(a) \quad U = 2X + 1.5Y \quad V = 2X - 2.5Y \\ m_U = 2m_X + 1.5m_Y = 2 \cdot 2 + 1.5 \cdot 5 = 11.5 \\ m_V = 2m_X - 2.5 \cdot 5 = -8.5$$

$$\sigma_U^2 = (2)^2 \sigma_X^2 + (1.5)^2 \sigma_Y^2 + 2(2)(1.5)C_{XY} = 4 \cdot 9 + 2.25 \cdot 4 + 6 \cdot 5 = 75$$

$$\sigma_V^2 = (2)^2 \sigma_X^2 + (-2.5)^2 \sigma_Y^2 + 2(2)(-2.5)C_{XY} = 4 \cdot 9 + 6.25 \cdot 4 - 10 \cdot 5 = 11$$

$$\begin{aligned} C_{UV} &= 2(2) \cdot \sigma_X^2 + (1.5)(-2.5)\sigma_Y^2 + 2(-2.5)C_{XY} + 2(1.5)C_{XY} \\ &= 4 \cdot 9 - 3.75 \cdot 4 - 5 \cdot 5 + 3 \cdot 5 = 11 \end{aligned}$$

$$\rho_{UV} = \frac{11}{\sqrt{11}\sqrt{75}} = \frac{11}{5\sqrt{11}\sqrt{3}} = \frac{1}{5}\sqrt{\frac{11}{3}} = 0.3829$$

(b) $U = aX + bY + m_1$

$$V = dY + m_2$$

$$m_U = a \cdot 2 + b \cdot 5 + m_1 = 0 \Rightarrow m_1 = -2a - 5b$$

$$m_V = d \cdot 5 + m_2 = 0 \Rightarrow m_2 = -5d$$

$$\sigma_U^2 = a^2 \sigma_X^2 + b^2 \sigma_Y^2 + 2abC_{XY} = 9a^2 + 4b^2 + 10ab = 1$$

$$\sigma_V^2 = d^2 \cdot \sigma_Y^2 = 4d^2 = 1 \quad \text{Solve for d to obtain } d=0.5$$

$$C_{UV} = bd\sigma_Y^2 + adC_{XY} = \frac{1}{2}b \cdot 4 + \frac{a}{2} \cdot 5 = 0 \quad \text{Solve for a in terms of b:}$$

$$a = -0.8b$$

Substitute into the other equation with a and b (the one with $\sigma_U^2 = 1$):

$$9(-0.8b)^2 + 4b^2 + 10(-0.8)b^2 = 1$$

$$(5.76 + 4 - 8)b^2 = 1$$

$$b = \frac{1}{\sqrt{1.76}} = 0.754$$

$$a = -0.8(0.754) = -0.603$$

$$d = 0.5$$

$$m_2 = 5 \cdot 0.5 = -2.5$$

$$m_1 = -2(-0.8b) - 5b = -3.4b = -2.56$$

(c) Since U and V are unit Gaussian and are uncorrelated their density is:

$$f_{UV}(u,v) = \{\exp[-(u^2 + v^2)/2]\} / \sqrt{2\pi}$$