

1. (a)

$$f_X(u) = \int_0^{2-u} f_{XY}(u, v) dv = \int_0^{2-u} 0.5 dv = 0.5v \Big|_0^{2-u} = 1 - 0.5u, \quad \text{for } 0 < u < 2$$

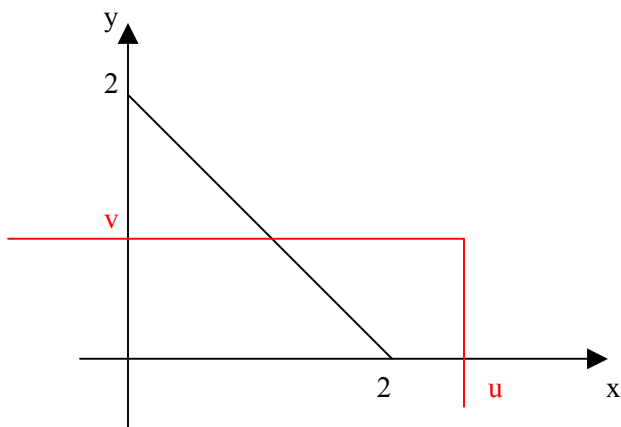
$$f_Y(v) = \int_0^{2-v} f_{XY}(u, v) du = \int_0^{2-v} 0.5 du = 1 - 0.5v, \quad \text{for } 0 < v < 2$$

(b) $f_{XY}(u, v) \neq f_X(u)f_Y(v)$, They are dependent!(c) $F_{XY}(u, v) = 0$, for $u < 0$ or $v < 0$
 $F_{XY}(u, v) = 1$, for $u > 2$ and $v > 2$

$$F_{XY}(u, v) = \int_0^u \int_0^v 0.5 dx dy = 0.5uv, \quad \text{for } u > 0 \text{ and } u + v < 2$$

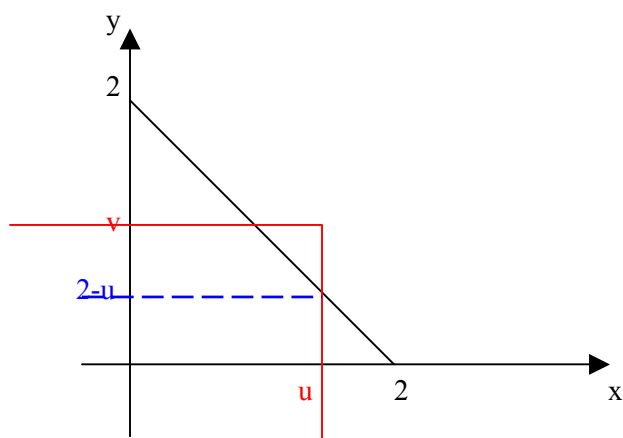
$$F_{XY}(u, v) = \int_0^v \int_0^{2-y} 0.5 dx dy = \int_0^v 0.5(2 - y) dy = v - \frac{v^2}{4}, \quad \text{for } u > 2 \text{ and } 0 < v < 2$$

See picture below of the region to be integrated (bound by the diagonal line and the red lines)



$$F_{XY}(u, v) = \int_0^u \int_0^{2-x} 0.5 dx dy = \int_0^u 0.5(2 - x) dx = u - \frac{u^2}{4}, \quad \text{for } 0 < u < 2 \text{ and } v > 2$$

Finally, for: $(u + v) > 2$ and $u < 2$ and $v < 2$ we get (based on the figure below – the area bounded by the red lines and the diagonal line):



$$\begin{aligned}
 F_{XY}(u, v) &= \int_0^{2-u} \int_0^u 0.5 dx dy + \int_{2-u}^v \int_0^{2-y} 0.5 dx dy \\
 &= 0.5u(2-u) + \int_{2-u}^v 0.5(2-y) dy = 0.5(2-u)u + \frac{u^2}{4} - \frac{(v-2)^2}{4} \\
 &= u - \frac{u^2}{4} - \frac{(v-2)^2}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad E\{XY\} &= \int_0^2 \int_0^{2-v} uv \cdot 0.5 du dv = \int_0^2 v \frac{(2-v)^2}{4} dv \\
 &= \int_0^2 v \left(1 - v + \frac{v^2}{4}\right) dv = \frac{2^2}{2} - \frac{2^3}{3} + \frac{2^4}{16} = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad E\{e^{-(X+Y)}\} &= \int_0^2 \int_0^{2-v} e^{-(u+v)} \cdot 0.5 du dv \\
 &= \int_0^2 0.5 e^{-v} [1 - e^{-(2-v)}] dv \\
 &= 0.5(1 - e^{-2}) - 0.5 \cdot 2e^{-2} = 0.5(1 - 3e^{-2}) = 0.297
 \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad E\{X\} &= E\{Y\} = \int_0^2 u(1 - 0.5u) du = \frac{4}{2} - \frac{8}{6} = \frac{2}{3} \\
 E\{X^2\} &= E\{Y^2\} = \int_0^2 u^2(1 - 0.5u) du = \frac{8}{3} - \frac{16}{8} = \frac{2}{3} \\
 \sigma_X^2 &= \sigma_Y^2 = E\{X^2\} - [E\{X\}]^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9} \\
 \rho_{XY} &= \frac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_X \sigma_Y} = \frac{\frac{1}{3} - \frac{4}{9}}{\frac{2}{9}} = -\frac{1}{2}
 \end{aligned}$$

2. Note that X and Y are independent with

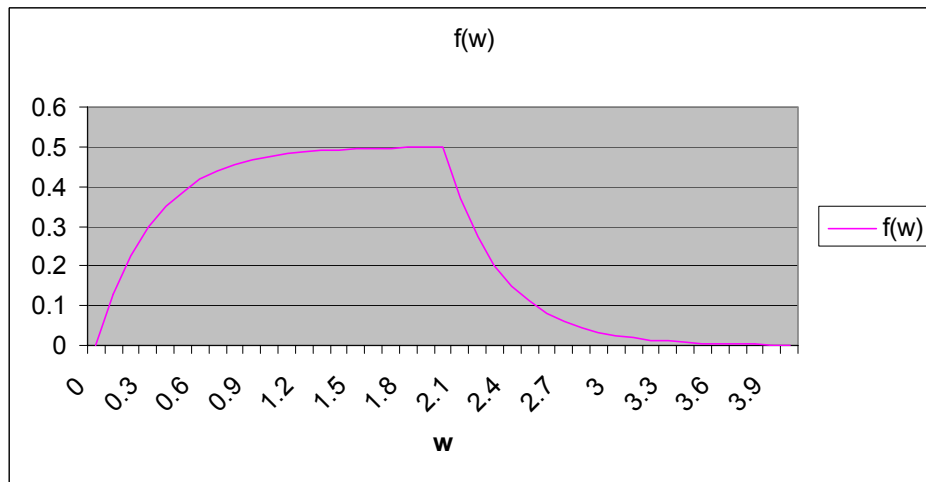
$$\begin{aligned} f_X(u) &= 0.5, & \text{for } 0 < u < 2 \\ f_Y(v) &= 3 \exp(-3v), & \text{for } 0 < v < \infty \end{aligned}$$

(a) $Z = X + Y$

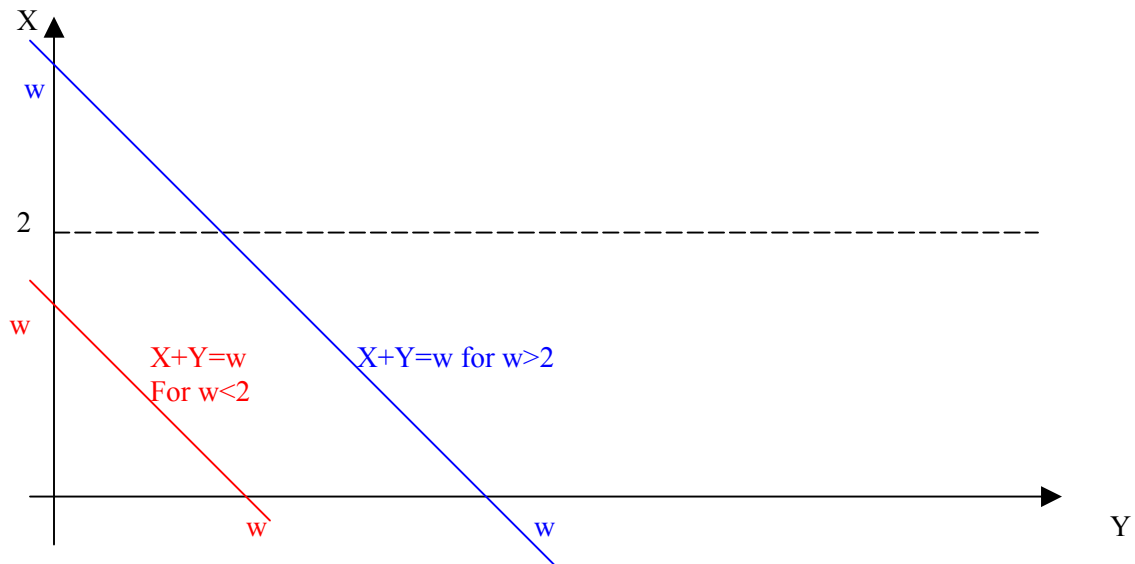
$$f_z(w) = \int_{-\infty}^{\infty} f_X(u) \cdot f_Y(w-u) du = 0, \quad \text{for } w < 0$$

$$f_z(w) = \int_0^w \frac{1}{2} \cdot 3e^{-3(w-u)} du = 1.5e^{-3w} \frac{e^{3u}}{3} \Big|_0^w = 0.5(1 - e^{-3w}), \quad \text{for } 0 < w < 2$$

$$\begin{aligned} f_z(w) &= \int_0^2 \frac{1}{2} \cdot 3e^{-3(w-u)} du = 1.5e^{-3w} \frac{e^{3u}}{3} \Big|_0^2 \\ &= 0.5(-1 + e^6)e^{-3w}, \quad \text{for } 2 < w < \infty \end{aligned}$$



Alternatively, we can first find the distribution and then differentiate to obtain the density function.



When $w < 2$ we use the area bounded by the red line and the X and Y axes to find $P\{(X+Y) \leq w\}$:

$$F_Z(w) = P\{(X+Y) \leq w\} = \int_0^w \int_0^{w-u} 1.5e^{-3v} dv du = \int_0^w 0.5(1 - e^{-3(w-u)}) du = 0.5w - e^{-3w} \frac{e^{3w} - 1}{6}$$

$$= 0.5w - (1 - e^{-3w})/6, \text{ for } 0 < w < 2$$

When $w > 2$ we use the area bounded by the blue line and the X and Y axes and $X=2$, to find $P\{(X+Y) \leq w\}$:

$$F_Z(w) = P\{(X+Y) \leq w\} = \int_0^2 \int_0^{w-u} 1.5e^{-3v} dv du = \int_0^2 0.5(1 - e^{-3(w-u)}) du = 1 - e^{-3w} \frac{e^6 - 1}{6}, \text{ for } 2 < w$$

Taking the derivative will yield the same two expressions for the density function we derived earlier.

$$(b) \quad E\{Z\} = E\{X + Y\} =$$

$$= E\{X\} + E\{Y\} = \int_0^2 \frac{1}{2} u du + \int_0^\infty v 3e^{-3v} dv = 1 + \frac{1}{3} = \frac{4}{3}$$

$$E\{Z^2\} = E\{(X + Y)^2\} = E\{X^2\} + E\{Y^2\} + 2E\{XY\}$$

$$E\{X^2\} = \int_0^2 u^2 \frac{1}{2} du = \frac{4}{3}$$

$$E\{Y^2\} = \int_0^\infty v^2 3e^{-3v} dv = 2 \cdot \frac{1}{3^2} = \frac{2}{9}$$

$$E\{XY\} = \int_0^2 \int_0^\infty uv \frac{1}{2} 3e^{-3v} dv du = E\{X\}E\{Y\} = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

$$\sigma_z^2 = E\{Z^2\} - [E\{Z\}]^2$$

$$= \frac{4}{3} + \frac{2}{9} + \frac{2}{3} - \left(\frac{4}{3}\right)^2 = \frac{4}{9} = \sigma_X^2 + \sigma_Y^2 \quad \text{as expected.}$$

$$(c) \quad E\{e^{-XY}\} = \int_0^2 \int_0^\infty e^{-uv} 1.5e^{-3v} dv du$$

$$= \int_0^2 \frac{1.5 du}{(3-u)} = 1.5 \ln\left(\frac{5}{3}\right) = 0.766$$

$$3. \quad Z = 2X + 1.5Y - 2.5$$

$$m_X = 2, \quad m_Y = 5, \quad \sigma_X = 3, \quad \sigma_Y = 2$$

$$m_Z = 2m_X + 1.5m_Y - 2.5 = 4 + 7.5 - 2.5 = 9$$

$$\sigma_Z^2 = (2)^2 \sigma_X^2 + (1.5)^2 \sigma_Y^2 + 2(2)(1.5)C_{XY} = 4 \cdot 9 + 2.25 \cdot 4 + 0 = 45$$

$$C_{XZ} = 2\sigma_X^2 + 1.5C_{XY} = 2 \cdot 9 + 0 = 18$$

$$\rho_{XZ} = \frac{18}{3 \cdot \sqrt{45}} = \frac{2}{\sqrt{5}} = 0.8944$$

$$C_{YZ} = 2C_{XY} + 1.5\sigma_Y^2 = 2 \cdot 0 + 1.5 \cdot 4 = 6 \quad \rho_{YZ} = \frac{6}{2\sqrt{45}} = \frac{1}{\sqrt{5}} = 0.4472$$