

$$\begin{aligned}
 \text{5-4. (a) } 1 &= \int_0^2 \int_0^1 C(2u+v) du dv = C \int_0^2 (u^2 + vu) \Big|_{u=0}^{u=1} dv \\
 &= C \int_0^2 [1+v] dv = C \frac{(1+v)^2}{2} \Big|_0^2 = C \left[\frac{9-1}{2} \right] = C4 \\
 C &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } f_X(u) &= \frac{1}{4} \int_0^2 (2u+v) dv = \frac{1}{4} \left[2uv + \frac{v^2}{2} \right] \Big|_{v=0}^{v=2} = \frac{1}{4} (4u+2) = \frac{1+2u}{2}, \\
 &\text{for } 0 \leq u \leq 1 \text{ and is zero elsewhere.}
 \end{aligned}$$

$$\begin{aligned}
 f_Y(v) &= \frac{1}{4} \int_0^1 (2u+v) du = \frac{1}{4} (u^2 + vu) \Big|_{u=0}^{u=1} = \frac{1+v}{4}, \\
 &\text{for } 0 \leq v \leq 2 \text{ and is zero elsewhere.}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } F_{XY}(u, v) &= 0, \quad \text{for } u < 0 \quad \text{or} \quad v < 0 \\
 &\text{For } 0 < u < 1 \quad \text{and} \quad 0 < v < 2 \text{ we have:}
 \end{aligned}$$

$$\begin{aligned}
 F_{XY}(u, v) &= \int_0^v \int_0^u f_{XY}(\alpha, \beta) d\alpha d\beta \\
 &= \frac{1}{4} \int_0^v \int_0^u (2\alpha + \beta) d\alpha d\beta = \frac{1}{4} \int_0^v [\alpha^2 + \beta\alpha]_{\alpha=0}^{\alpha=u} d\beta \\
 &= \frac{1}{4} \int_0^v (u^2 + \beta u) d\beta = \frac{1}{4} \left[u^2 \beta + u \frac{\beta^2}{2} \right] \Big|_0^v = \frac{1}{4} \left[u^2 v + \frac{uv^2}{2} \right] \\
 &= \frac{uv}{4} \left(u + \frac{v}{2} \right), \quad \text{for } 0 < u < 1 \quad \text{and} \quad 0 < v < 2
 \end{aligned}$$

For $1 \leq u$, and $0 < v < 2$ we have:

$$F_{XY}(u, v) = F_{XY}(1, v) = \frac{1}{4} \left(v + \frac{v^2}{2} \right)$$

For $0 < u < 1$, and $2 \leq v$ we have

$$F_{XY}(u, v) = F_{XY}(u, 2) = \frac{1}{4} (2u^2 + 2u) = \frac{u^2 + u}{2}$$

Finally, for $1 \leq u$, and $2 \leq v$

$$F_{XY}(u, v) = 1$$

$$(d) \quad F_X(u) = \begin{cases} 0, & \text{for } u < 0 \\ F_{XY}(u, 2) = \frac{u^2 + u}{2}, & \text{for } 0 \leq u < 1 \\ 1.0, & \text{for } 1 \leq u \end{cases}$$

$$F_Y(v) = \begin{cases} 0, & \text{for } v < 0 \\ F_{XY}(1, v) = \frac{1}{8}(2v + v^2), & \text{for } 0 \leq v < 2 \\ 1.0, & \text{for } 2 \leq v \end{cases}$$

$$(e) \quad P\{0.25 < X < 0.5, 1 < Y < 1.5\} =$$

$$F_{XY}(0.5, 1.5) - F_{XY}(0.5, 1) - F_{XY}(0.25, 1.5) + F_{XY}(0.25, 1) = 0.0625$$

$$P\{0.5 < X < 1.2, 1 < Y < 1.5\} = P\{0.5 < X < 1.0, 1 < Y < 1.5\}$$

$$= F_{XY}(1.0, 1.5) - F_{XY}(1.0, 1.0) - F_{XY}(0.5, 1.5) + F_{XY}(0.5, 1) = 0.172$$

$$P\{0.25 < X < 0.5, 1 < Y < 2.5\} = P\{0.25 < X < 0.5, 1 < Y < 2\} =$$

$$= F_{XY}(0.5, 2) - F_{XY}(0.5, 1) - F_{XY}(0.25, 2) + F_{XY}(0.25, 1) = 0.1406$$

5-10. We first find the value of C:

$$1 = \int_0^\infty \int_{0.5u}^{2u} C \exp\{-2(u+v)\} dv du =$$

$$= C \int_0^\infty \exp(-2u) \{0.5 \exp[-2(0.5u)] - 0.5 \exp[-2(2u)]\} du =$$

$$= C(0.5) \{ (1/3) - (1/6) \} = C(1/12); \text{ Hence } C=12$$

$$(a) \quad f_X(u) = \int_{0.5u}^{2u} 12 \exp\{-2(u+v)\} dv = 12 \exp(-2u) (0.5) \{ \exp[-2(0.5u)] - \exp[-2(2u)] \}$$

$$= 6 \{ \exp(-3u) - \exp(-6u) \}$$

The expression for $f_Y(v) = 6 \{ \exp(-3v) - \exp(-6v) \}$ is identical due to the symmetry of the problem (including the region).

6-2. (a) $f_X(u) = \int_0^1 (u+v) dv = \left(uv + \frac{v^2}{2} \right) \Big|_{v=0}^1 = \frac{1}{2} + u, \quad \text{for } 0 \leq u \leq 1,$

and is zero elsewhere

$$f_Y(v) = \int_0^1 (u+v) du = \frac{1}{2} + v, \quad \text{for } 0 \leq v \leq 1, \text{ and is zero elsewhere}$$

$f_{XY}(u, v) \neq f_X(u) f_Y(v)$. They are dependent.

$$\begin{aligned}
\text{(b)} \quad E\{g(X, Y)\} &= E\{(X + Y)\} = \int_0^1 \int_0^1 (u + v)(u + v) du dv \\
&= \int_0^1 \left. \frac{(u + v)^3}{3} \right|_{u=0}^1 dv = \int_0^1 \left[\frac{(1 + v)^3 - v^3}{3} \right] dv \\
&= \left. \frac{(1 + v)^4 - v^4}{12} \right|_0^1 = \frac{2^4 - 1}{12} - \frac{1 - 0}{12} = \frac{7}{6}
\end{aligned}$$

Check: $E\{X\} = \int_0^1 u \left(\frac{1}{2} + u \right) du = \left[\frac{u^2}{4} + \frac{u^3}{3} \right]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} = E\{Y\}$

$$E\{X + Y\} = E\{X\} + E\{Y\} = \frac{7}{6}$$

$$\begin{aligned}
\text{(c)} \quad E\{g(X, Y)\} &= E\{XY\} = \int_0^1 \int_0^1 uv(u + v) du dv \\
&= \int_0^1 v \left[\frac{u^3}{3} + \frac{vu^2}{2} \right]_{u=0}^1 dv = \int_0^1 v \left(\frac{1}{3} + \frac{v}{2} \right) dv = \left[\frac{v^2}{6} + \frac{v^3}{6} \right]_0^1 = \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}$$

$$\text{(d)} \quad E\{(X + Y)^2\} = \int_0^1 \int_0^1 (u + v)^2 (u + v) du dv$$

$$= \int_0^1 \left[\frac{(u + v)^4}{4} \right]_{u=0}^1 dv = \int_0^1 \frac{(1 + v)^4 - v^4}{4} dv = \left. \frac{(1 + v)^5 - v^5}{20} \right|_0^1 = \frac{2^5 - 1 - 1}{20} = 1.6$$

$$\begin{aligned}
\text{(e)} \quad E\{X^2 Y^2\} &= \int_0^1 \int_0^1 u^2 v^2 (u + v) du dv \\
&= \int_0^1 v^2 \left(\frac{u^4}{4} + \frac{vu^3}{3} \right)_{u=0}^1 dv = \int_0^1 v^2 \left(\frac{1}{4} + \frac{v}{3} \right) dv \\
&= \left[\frac{v^3}{12} + \frac{v^4}{12} \right]_0^1 = \frac{1}{6}
\end{aligned}$$