

Solution of Regular Problems from Chapter 5:

5-3. Y takes 3 values $P\{Y=6\}=P\{Y=24\}=0.25$ $P\{Y=12\}=0.5$

The loads are 3, 6, 12, and 24 with equal probability of 0.25. For each load we compute all the values of $X = \frac{\text{voltage}}{\text{load}}$ and assign the following probability to it:
 $= 0.25 \cdot P\{Y = \text{voltage}\}.$

(a) The results are summarized in the following Table:

X \ Y	0.25	0.5	1	2	4	8	P{Y= }
6	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$			0.25
12		$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		0.5
24			$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0.25
P{X= }	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$	1.0

The center of the Table shows the joint probability mass function of X and Y. The marginal probability mass functions of X and Y are shown at right and at the bottom of the Table.

(b) The joint distribution function is obtained by adding the probabilities in the table:

$$F_{XY}(u,v) = \begin{cases} 0, & u < 0.25 \text{ or } v < 6 \\ 1/16, & 0.25 \leq u < 0.5, 6 \leq v < 12 \\ 1/8, & 0.5 \leq u < 1, 6 \leq v < 12 \\ 1/4, & 0.5 \leq u < 1, 12 \leq v < 24 \\ 3/16, & 1 \leq u < 2, 6 \leq v < 12 \\ 7/16, & 1 \leq u < 2, 12 \leq v < 24 \\ 1/2, & 1 \leq u < 2, 24 \leq v \\ 1/4, & 2 \leq u < 4, 6 \leq v < 12 \\ 5/8, & 2 \leq u < 4, 12 \leq v < 24 \\ 3/4, & 2 \leq u < 4, 24 \leq v \\ 1/4, & 4 \leq u < 8, 6 \leq v < 12 \\ 3/4, & 4 \leq u < 8, 12 \leq v < 24 \\ 15/16, & 4 \leq u < 8, 24 \leq v \\ 1/4, & 8 \leq u, 6 \leq v < 12 \\ 3/4, & 8 \leq u, 12 \leq v < 24 \\ 1.0, & 8 \leq u, 24 \leq v \end{cases}$$

$$F_X(u) = \begin{cases} 0, & u < 0.25 \\ 1/16, & 0.25 \leq u < 0.5 \\ 1/4, & 0.5 \leq u < 1 \\ 1/2, & 1 \leq u < 2 \\ 3/4, & 2 \leq u < 4 \\ 15/16, & 4 \leq u < 8 \\ 1.0, & 8 \leq u \end{cases}$$

$$F_Y(v) = \begin{cases} 0, & v < 6 \\ 0.25, & 6 \leq v < 12 \\ 0.75, & 12 \leq v < 24 \\ 1.0, & 24 \leq v \end{cases}$$

$$(c) P\{X=1, Y \leq 12\} = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} = \frac{7}{16} = F_{XY}(1,12)$$

$$\begin{aligned} P\{0.5 < X \leq 2, 10 < Y \leq 12\} &= F_{XY}(2,12) - F_{XY}(2,10) - F_{XY}(0.5,12) + F_{XY}(0.5,10) \\ &= \frac{5}{8} - \frac{1}{4} - \frac{1}{4} + \frac{1}{8} = \frac{1}{4} \end{aligned}$$

5-5. (a) $F_{XY}(u, v) = u^2(1 - e^{-2v})$, $0 < u < 1$, $0 < v < \infty$
 $F_{XY}(u, v) = 0$ for $u < 0$ or $v < 0$
 $F_{XY}(u, v) = F_{XY}(1, v) = 1 - e^{-2v}$ for $u < 0$, $0 < v < \infty$

$$(b) F_X(u) = \begin{cases} 0, & \text{for } u < 0 \\ F_{XY}(u, \infty) = u^2, & \text{for } 0 < u < 1 \\ 1.0, & \text{for } 1 \leq u \end{cases}$$

$$F_Y(v) = \begin{cases} 0, & \text{for } v < 0 \\ F_{XY}(1, v) = 1 - e^{-2v}, & \text{for } 0 < v < \infty \end{cases}$$

(c) $f_{XY}(u, v) = 2u \cdot 2e^{-2v} = 4ue^{-2v}$, for $0 < u < 1$, $0 < v < \infty$
and is zero elsewhere

(d) $f_X(u) = 2u$, for $0 < u < 1$ and is zero elsewhere

$f_Y(v) = 2e^{-2v}$, for $0 < v < \infty$ and is zero elsewhere

(e) $P\{0.5 < X < 0.8, 1 < Y < 2\} =$
 $F_{XY}(0.8, 2) - F_{XY}(0.8, 1) - F_{XY}(0.5, 2) + F_{XY}(0.5, 1) = 0.0456$

$$\begin{aligned} P\{0.5 < X < 1.5, 1 < Y < 2\} &= P\{0.5 < X < 1, 1 < Y < 2\} \\ &= F_{XY}(1, 2.0) - F_{XY}(1, 1) - F_{XY}(0.5, 2) + F_{XY}(0.5, 1) = 0.0878 \end{aligned}$$

5-4. (a) $1 = \int_0^2 \int_0^1 C(2u+v) du dv = C \int_0^2 (u^2 + vu) \Big|_{u=0}^{u=1} dv$
 $= C \int_0^2 [1+v] dv = C \frac{(1+v)^2}{2} \Big|_0^2 = C \left[\frac{9-1}{2} \right] = C4$
 $C = \frac{1}{4}$

(b) $f_X(u) = \frac{1}{4} \int_0^2 (2u+v) dv = \frac{1}{4} \left[2uv + \frac{v^2}{2} \right] \Big|_{v=0}^{v=2} = \frac{1}{4} (4u+2) = \frac{1+2u}{2},$
for $0 \leq u \leq 1$ and is zero elsewhere.

$f_Y(v) = \frac{1}{4} \int_0^1 (2u+v) du = \frac{1}{4} (u^2 + vu) \Big|_{u=0}^{u=1} = \frac{1+v}{4},$
for $0 \leq v \leq 2$ and is zero elsewhere.

(c) $F_{XY}(u, v) = 0,$ for $u < 0$ or $v < 0$
For $0 < u < 1$ and $0 < v < 2$ we have:

$$\begin{aligned} F_{XY}(u, v) &= \int_0^v \int_0^u f_{XY}(\alpha, \beta) d\alpha d\beta \\ &= \frac{1}{4} \int_0^v \int_0^u (2\alpha + \beta) d\alpha d\beta = \frac{1}{4} \int_0^v [\alpha^2 + \beta\alpha]_{\alpha=0}^{\alpha=u} d\beta \\ &= \frac{1}{4} \int_0^v (u^2 + \beta u) d\beta = \frac{1}{4} \left[u^2 \beta + u \frac{\beta^2}{2} \right] \Big|_0^v = \frac{1}{4} \left[u^2 v + \frac{uv^2}{2} \right] \\ &= \frac{uv}{4} \left(u + \frac{v}{2} \right), \quad \text{for } 0 < u < 1 \text{ and } 0 < v < 2 \end{aligned}$$

For $1 \leq u$, and $0 < v < 2$ we have:

$$F_{XY}(u, v) = F_{XY}(1, v) = \frac{1}{4} \left(v + \frac{v^2}{2} \right)$$

For $0 < u < 1$, and $2 \leq v$ we have

$$F_{XY}(u, v) = F_{XY}(u, 2) = \frac{1}{4} (2u^2 + 2u) = \frac{u^2 + u}{2}$$

Finally, for $1 \leq u$, and $2 \leq v$

$$F_{XY}(u, v) = 1$$

$$(d) \quad F_X(u) = \begin{cases} 0, & \text{for } u < 0 \\ F_{XY}(u, 2) = \frac{u^2 + u}{2}, & \text{for } 0 \leq u < 1 \\ 1.0, & \text{for } 1 \leq u \end{cases}$$

$$F_Y(v) = \begin{cases} 0, & \text{for } v < 0 \\ F_{XY}(1, v) = \frac{1}{8}(2v + v^2), & \text{for } 0 \leq v < 2 \\ 1.0, & \text{for } 2 \leq v \end{cases}$$

$$(e) \quad P\{0.25 < X < 0.5, 1 < Y < 1.5\} =$$

$$F_{XY}(0.5, 1.5) - F_{XY}(0.5, 1) - F_{XY}(0.25, 1.5) + F_{XY}(0.25, 1) = 0.0625$$

$$P\{0.5 < X < 1.2, 1 < Y < 1.5\} = P\{0.5 < X < 1.0, 1 < Y < 1.5\}$$

$$= F_{XY}(1.0, 1.5) - F_{XY}(1.0, 1.0) - F_{XY}(0.5, 1.5) + F_{XY}(0.5, 1) = 0.172$$

$$P\{0.25 < X < 0.5, 1 < Y < 2.5\} = P\{0.25 < X < 0.5, 1 < Y < 2\} =$$

$$= F_{XY}(0.5, 2) - F_{XY}(0.5, 1) - F_{XY}(0.25, 2) + F_{XY}(0.25, 1) = 0.1406$$