

Solution of Regular Problems from Chapter 4:

3. $X = \#$ of transmissions until success. $p = 0.6 = \text{Prob. of Collision}$.
The distribution is geometric with probability of success 0.4 (no collision).

$$P\{X = k\} = (0.6)^{k-1} 0.4 \quad k = 1, 2, 3, \dots$$

The number of retransmission is one less than the number of transmissions!

- a. Average number of Retransmissions $= E\{X\} - 1 = \frac{1}{0.4} - 1 = 1.5$
 b. The average number of total transmission is $E\{X\} = 1/0.4 = 2.5$

4. $\lambda = 2/\text{sec}$.

First we have: $T = 2 \text{ sec}$. $\lambda T = 4$, so that $E\{X\} = \lambda T = 4$
 $\sigma_X = \sqrt{\lambda T} = \sqrt{4} = 2 = 0.5E\{X\}$

Second $T = 1 \text{ min} = 60 \text{ sec}$. $\lambda T = 120$, so that $E\{X\} = 120$
 $\sigma_X = \sqrt{\lambda T} = \sqrt{120} = 10.95 = 0.09E\{X\}$

Third: $T = 10 \text{ min} = 600 \text{ sec}$. $\lambda T = 1200$, so that $E\{X\} = 1200$
 $\sigma_X = \sqrt{\lambda T} = \sqrt{1200} = 34.64 = 0.029E\{X\}$

The standard deviation as a fraction of the mean becomes smaller as T becomes larger, so we should be closer to the mean.

5. $f_X(u) = e^{-2|u|}$

$Y = g(X)$ is shown in Figure 5(a)

$$F_X(u) = \begin{cases} 0.5\exp(2u), & \text{for } u < 0 \\ 1 - 0.5\exp(-2u), & \text{for } 0 \leq u \end{cases}$$

(a) $Y = \text{Mixed random variable}$. It has jumps at -10 , 0 , and $+10$

Distribution of Y :

$$F_Y(u) = 0, \quad u < -10$$

For $-10 \leq u < 0$ we have

$$\begin{aligned} F_Y(u) &= P\{Y \leq u\} = P\{5(X + 0.1) \leq u\} \\ &= P\left\{X \leq \frac{u}{5} - 0.1\right\} = F_X(0.2u - 0.1) \\ &= 0.5e^{2(0.2u - 0.1)} = 0.5e^{0.4(u - 0.5)}, \quad -10 \leq u < 0 \end{aligned}$$

For $0 \leq u < 10$ we have

$$\begin{aligned}
F_Y(u) &= P\{Y \leq u\} = P\{5(X - 0.1) \leq u\} \\
&= P\{X \leq \frac{u}{5} + 0.1\} = F_X(0.2u + 0.1) \\
&= 1 - 0.5e^{-2(0.2u+0.1)} = 1 - 0.5e^{-0.4(u+0.5)}, & 0 \leq u < 10 \\
F_Y(u) &= 1, & 10 \leq u
\end{aligned}$$

The distribution is shown in Figure 5(b) over the entire range. Figures 5(c), 5(d), and 5(e) show the distribution near -10 , 0 , and $+10$ to show the jumps at these values.

$$\begin{aligned}
P\{Y = -10\} &= F_Y(-10) - F_Y(-10-) = 0.5e^{0.4(-10.5)} - 0 \\
&= 0.5e^{-4.2} = 0.0075
\end{aligned}$$

$$\begin{aligned}
P\{Y = +10\} &= 1 - F_Y(+10-) = 0.5e^{-0.4(10+0.5)} = 0.0075 \\
P\{Y = 0\} &= F_Y(0) - F_Y(0-) = 1 - 0.5e^{-0.4(0+0.5)} - 0.5e^{0.4(0-0.5)} \\
&= 1 - e^{-0.2} = 0.18
\end{aligned}$$

$$\begin{aligned}
(b) \quad E\{Y\} &= \int_{-\infty}^{\infty} g(u)f_X(u)du = \int_{-\infty}^{-2.1} (-10)e^{2u} du \\
&+ \int_{-2.1}^{-0.1} 5(u+0.1)e^{2u} du + \int_{-0.1}^{0.1} 0e^{-2|u|} du + \int_{0.1}^{2.1} 5(u-0.1)e^{-2u} du + \int_{2.1}^{\infty} 10e^{-2u} du = 0
\end{aligned}$$

Since the mean is zero, $\text{var}(Y) = E\{Y^2\}$

$$\begin{aligned}
\sigma_Y^2 &= E\{Y^2\} = E\{[g(x)]^2\} = \int_{-\infty}^{\infty} [g(u)]^2 f_X(u) du \\
&= \int_{-\infty}^{-2.1} (-10)^2 e^{2u} du + \int_{-2.1}^{-0.1} [(5(u+0.1))]^2 e^{2u} du + \int_{-0.1}^{0.1} 0 \cdot e^{-2|u|} du \\
&+ \int_{0.1}^{2.1} [5(u-0.1)]^2 e^{-2u} du + \int_{2.1}^{\infty} (10)^2 e^{-2u} du \\
&= 2 \left\{ \int_{0.1}^{2.1} [5(u-0.1)]^2 e^{-2u} du + \int_{2.1}^{\infty} 100e^{-2u} du \right\} \\
&= 2 \left\{ \frac{25}{(-2)} [(u-0.1)^2 + (u-0.1) + 0.5] e^{-2u} \Big|_{0.1}^{2.1} + \frac{100}{(-2)} e^{-2u} \Big|_{2.1}^{\infty} \right\} \\
&= 25[e^{-0.2} 0.5 - e^{-4.2} (4 + 2 + 0.5)] + 100e^{-4.2} \\
&= 12.5e^{-0.2} - 62.5e^{-4.2} = 9.297
\end{aligned}$$

6. $Y = 0.5 X^2$ It is shown in Figure 6(a).

(a) $F_Y(u) = P\{Y \leq u\} = P\{0.5X^2 \leq u\}$ This is shown for $u > 0$ in Figure 6(a) also.

For $u \leq 0$ $F_Y(u) = 0$, since no value of X yields negative values of Y .

For $u > 0$ we have as shown in Figure 6(a):

$$F_Y(u) = P\{-\sqrt{2u} \leq X \leq \sqrt{2u}\} = F_X(\sqrt{2u}) - F_X(-\sqrt{2u})$$

$$= 1 - 0.5e^{-2\sqrt{2}u} - 0.5e^{-2\sqrt{2}u} = 1 - e^{-2\sqrt{2}u}$$

For $u > 0$ the density is equal to:

$$f_Y(u) = \frac{d}{du} F_Y(u) = 2e^{-2\sqrt{2}u} \cdot \frac{\sqrt{2}}{2\sqrt{u}} = \sqrt{\frac{2}{u}} e^{-2\sqrt{2}u}, \quad \text{for } u > 0$$

The density and distribution functions are shown in Figure 6(b) and 4(c).

$$(b) \quad E\{Y\} = E\{g(X)\} = E\{0.5X^2\}$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} 0.5u^2 f_Y(u) du = \int_{-\infty}^{\infty} 0.5u^2 e^{-2|u|} du \\ &= \int_0^{\infty} u^2 e^{-2u} du = -\frac{1}{2} [u^2 + u + 0.5] e^{-2u} \Big|_0^{\infty} \\ &= \frac{1}{4} e^{-2 \cdot 0} = 0.25 \end{aligned}$$

$$7. \quad E\{X\} = m_X = 0.5, \quad \sigma_X^2 = 9, \quad \sigma_X = 3$$

$$Y = 5X - 1.5$$

(a) Linear operations:

$$m_Y = 5m_X - 1.5 = 5 \cdot 0.5 - 1.5 = 1.0$$

$$\sigma_Y^2 = (5)^2 \sigma_X^2 = 25 \cdot 9 = 225$$

$$\sigma_Y = 5 \cdot \sigma_X = 15$$

$$(b) \quad \text{If } X \text{ is Gaussian, then } Y \text{ is also Gaussian:} \quad f_Y(u) = \frac{\exp\left\{-\frac{(u-1)^2}{2(225)}\right\}}{\sqrt{2\pi}15}$$

Figures for Problem 5:

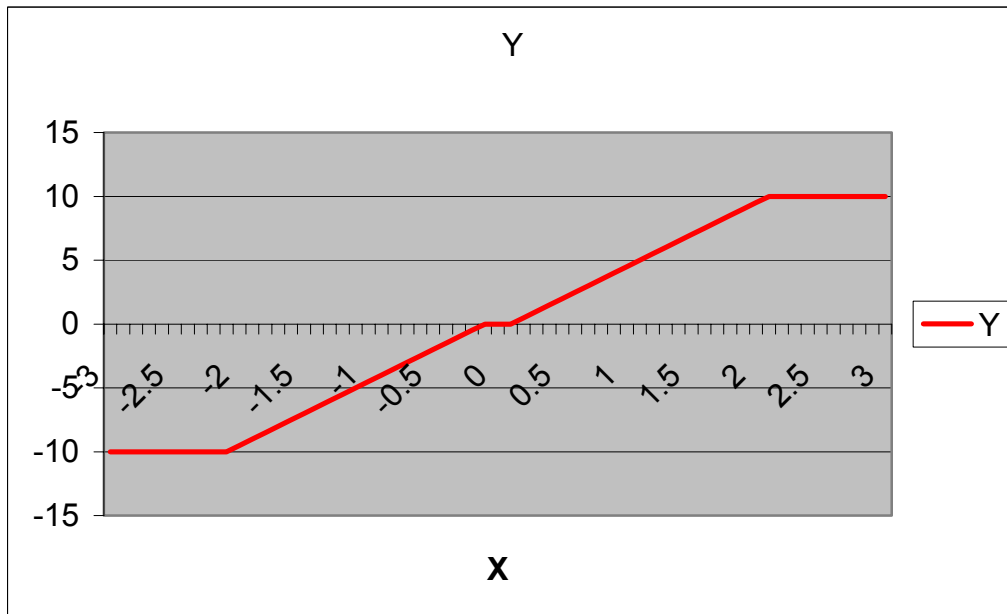


Figure 5(a) $Y=g(X)$

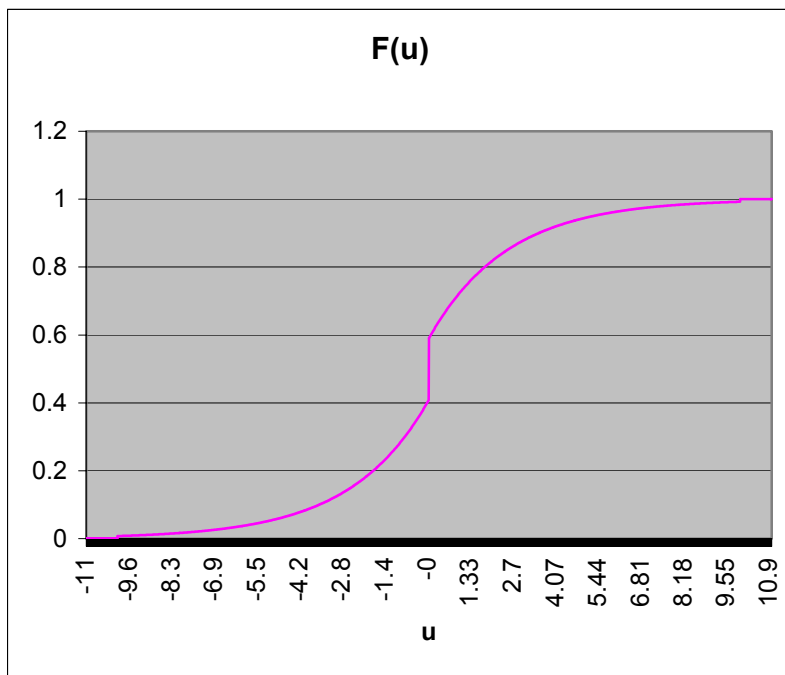


Figure 5(b) $F_Y(u)$ over its entire range

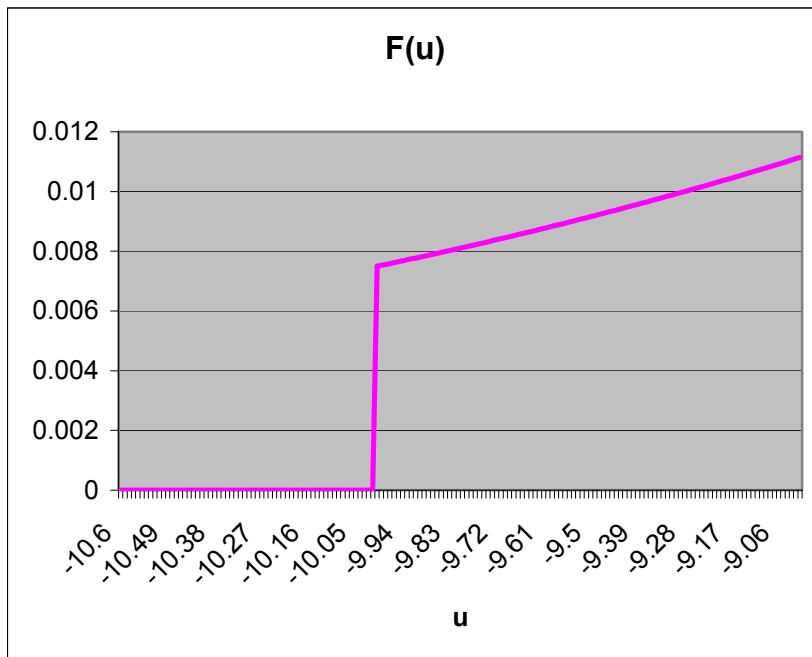


Figure 5(c) $F_Y(u)$ near $u = -10$ to show the jump at $Y = -10$

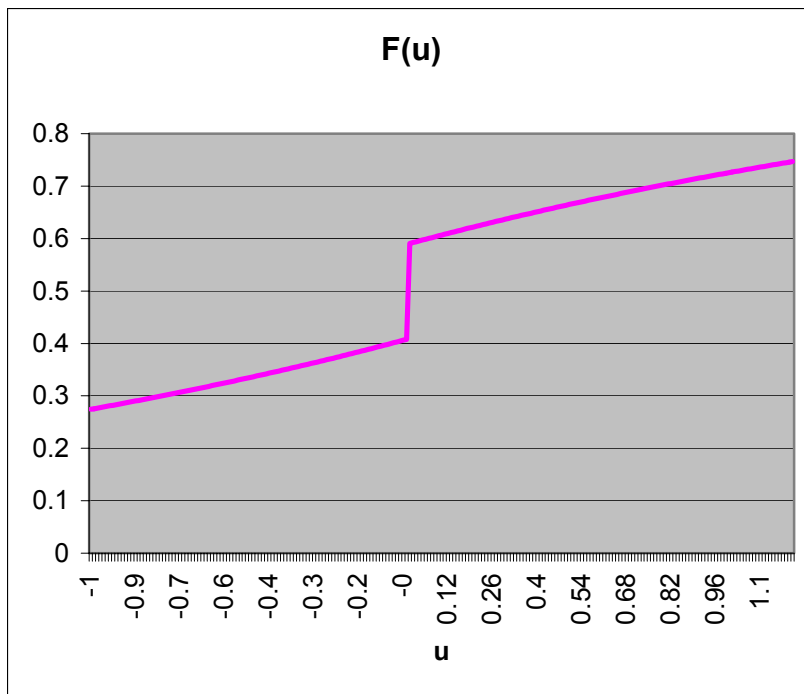


Figure 5(d) $F_Y(u)$ near $u=0$ to show the jump at $Y=0$

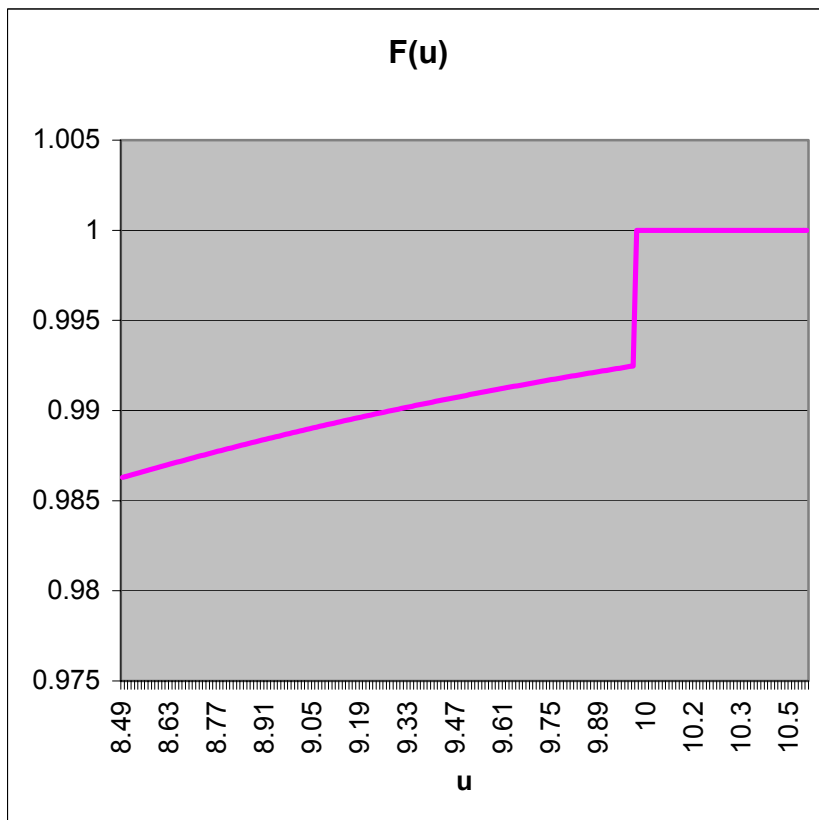


Figure 5(e) $F_Y(u)$ near $u=10$ to show the jump at $Y=10$

Figures for Problem 6:

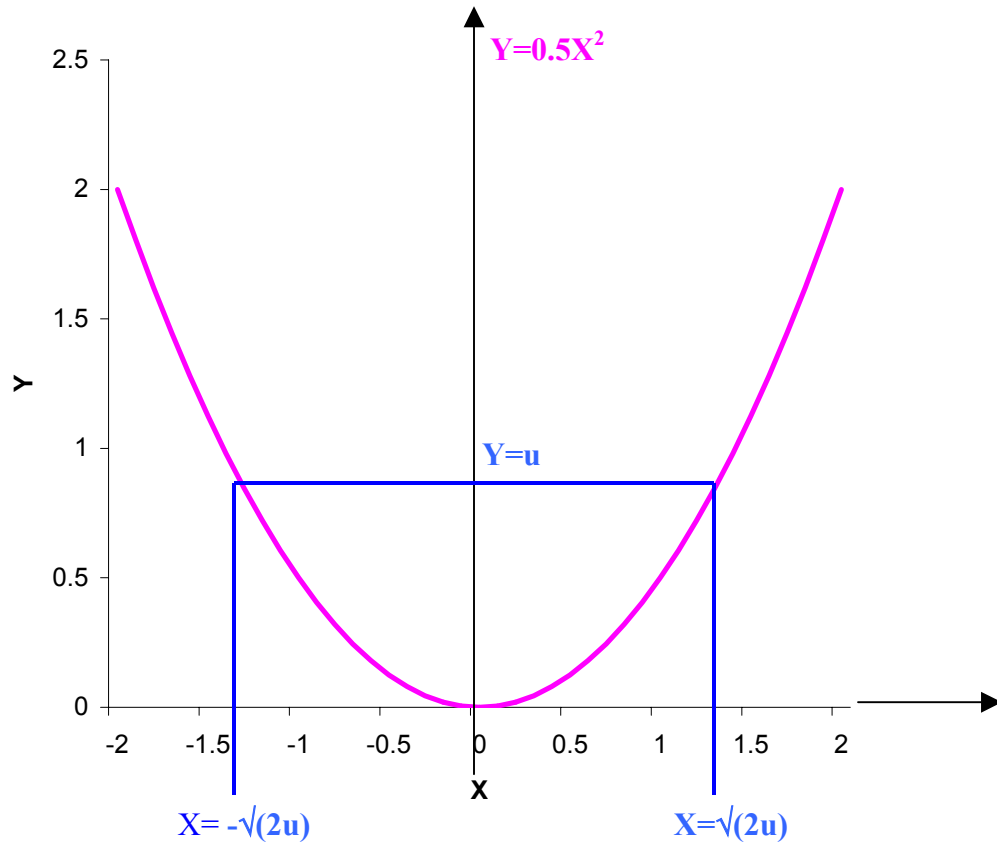


Figure 6(a) Illustrating $Y=g(X)$ and how $\{g(X) \leq u\}$ translates into $\{-\sqrt{(2u)} \leq X \leq \sqrt{(2u)}\}$

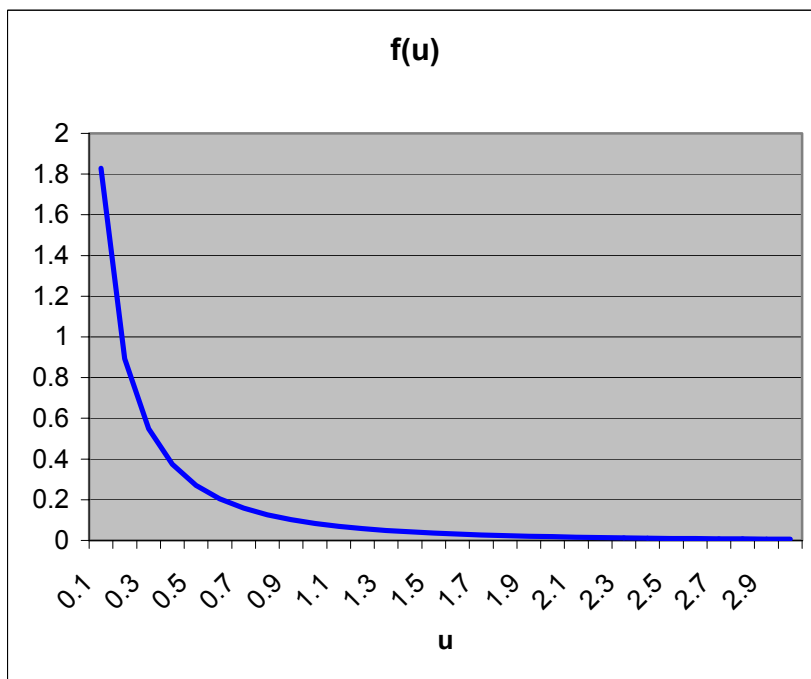


Figure 6(b) The density function of Y

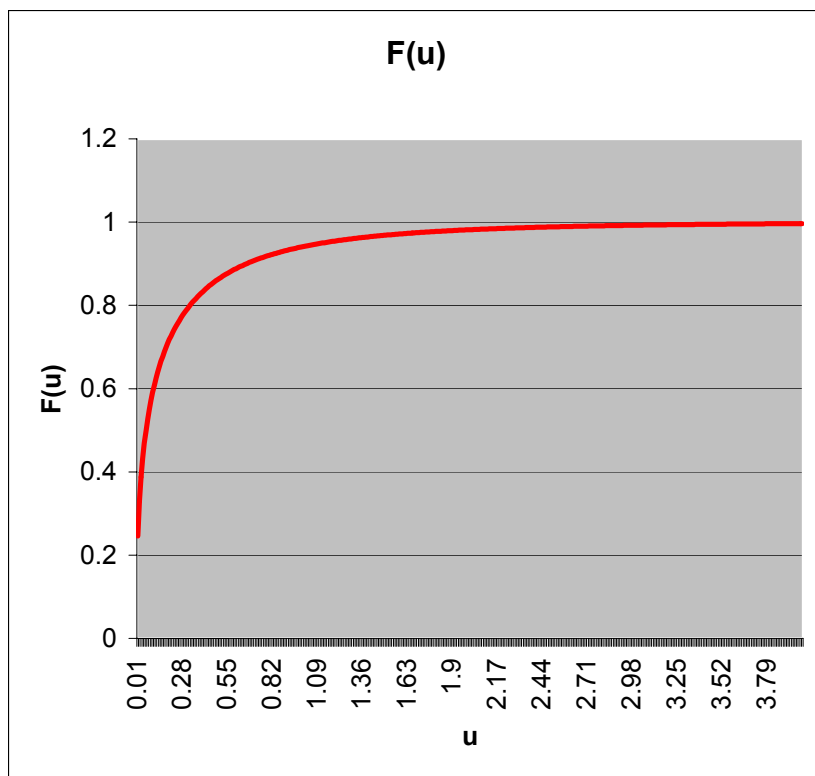


Figure 6(c) The distribution function of Y