

Solution of Regular Problems from Chapter 3:**3-3.** For System (a) only:**a.**

Event	X
$F_1 \cap F_2 \cap F_3$	0
$F_1 \cap \bar{F}_2 \cap F_3$	0
$F_1 \cap F_2 \cap \bar{F}_3$	0
$F_1 \cap \bar{F}_2 \cap \bar{F}_3$	0

Event	X
$\bar{F}_1 \cap F_2 \cap F_3$	0
$\bar{F}_1 \cap \bar{F}_2 \cap F_3$	6
$\bar{F}_1 \cap F_2 \cap \bar{F}_3$	4
$\bar{F}_1 \cap \bar{F}_2 \cap \bar{F}_3$	10

b. $P\{X = 0\} = P\{F_1\} + P\{\bar{F}_1 \cap F_2 \cap F_3\} = p_1 + (1 - p_1)p_2p_3 = 0.0298$

$$P\{X = 4\} = P\{\bar{F}_1 \cap F_2 \cap \bar{F}_3\} = (1 - p_1)p_2(1 - p_3) = 0.0792$$

$$P\{X = 6\} = P\{\bar{F}_1 \cap \bar{F}_2 \cap F_3\} = (1 - p_1)(1 - p_2)p_3 = 0.1782$$

$$P\{X = 10\} = p\{\bar{F}_1 \cap \bar{F}_2 \cap \bar{F}_3\} = (1 - p_1)(1 - p_2)(1 - p_3) = 0.7128$$

$$F_X(u) = \begin{cases} 0, & u < 0 \\ 0.0298, & 0 \leq u < 4 \\ 0.109, & 4 \leq u < 6 \\ 0.1872, & 6 \leq u < 10 \\ 1, & 10 \leq u \end{cases}$$

3-4. It is easier to consider as events, the number of switches that are ON from 0 to 4. Then we can use the Bernoulli trials to determine the appropriate probabilities.

# of Switches ON	X	Probabilities
0	0	$\binom{4}{0} p^0 (1 - p)^4 = (0.6)^4 = 0.1296$
1	5	$\binom{4}{1} p^1 (1 - p)^3 = 4(0.4)(0.6)^3 = 0.3456$
2	10	$\binom{4}{2} p^2 (1 - p)^2 = 6(0.4)^2 (0.6)^2 = 0.3456$
3	15	$\binom{4}{3} p^3 (1 - p)^1 = 4(0.4)^3 (0.6) = 0.1536$
4	20	$\binom{4}{4} p^4 (1 - p)^0 = (0.4)^4 = 0.0256$

The last two columns define the probability mass function of X

$$F_X(u) = \begin{cases} 0, & u < 0 \\ 0.1296, & 0 \leq u < 5 \\ 0.4752, & 5 \leq u < 10 \\ 0.8208, & 10 \leq u < 15 \\ 0.9744, & 15 \leq u < 20 \\ 1.0 & 20 \leq u \end{cases}$$

3-5. X takes only 5 values 18, 19, 20, 21, 22. Since we assume rounding-off we have:

$$P\{X = 18\} = P\{X = 22\} = \frac{0.5}{4} = 0.125$$

$$P\{X = 19\} = P\{X = 20\} = P\{X = 21\} = \frac{1}{4} = 0.25$$

$$F_X(u) = \begin{cases} 0 & u < 18 \\ 0.125, & 18 \leq u < 19 \\ 0.375, & 19 \leq u < 20 \\ 0.625, & 20 \leq u < 21 \\ 0.875, & 21 \leq u < 22 \\ 1.0 & 22 \leq u \end{cases}$$

3-6. (a) Valid – non-decreasing and continuous from the right.

$$P\{0 < X < 1\} = F_X(1-) - F_X(0) = 0.2(1+1) - 0.2(0+1) = 0.2$$

$$P\{0 < X \leq 1\} = F_X(1) - F_X(0) = 0.5 - 0.2(0+1) = 0.3$$

$$P\{1 < X < 2\} = F_X(2-) - F_X(1) = 0.5 - 0.5 = 0$$

$$P\{1 \leq X < 2\} = F_X(2-) - F_X(1-) = 0.5 - 0.2(1+1) = 0.1$$

$$P\{1 \leq X \leq 2\} = F_X(2) - F_X(1-) = 0.2(2+1) - 0.2(1+1) = 0.2$$

$$P\{X = 2\} = F_X(2) - F_X(2-) = 0.2(2+1) - 0.5 = 0.1$$

(b) Not valid, since it is not continuous from the right.

(c) Not valid, since it rises to 2 at $u = 2$ and **decreases** back to 1.

(d) Valid and is continuous with no jumps.

$$P\{0 < X < 1\} = P\{0 < X \leq 1\} = F_X(1) - F_X(0) = \frac{1}{2}$$

$$P\{1 < X < 2\} = P\{1 \leq X < 2\} = P\{1 \leq X \leq 2\} = F_X(2) - F_X(1) = 1 - \frac{1}{2} = 0.5$$

$$P\{X = 2\} = 0 \text{ No jumps}$$

3-7. (a) Not valid – negative in the range $1 < u < 2$

(b) Valid

$$1 = \int_0^1 C \sin(\pi u) du = -\frac{C}{\pi} \cos(\pi u) \Big|_0^1 = \frac{2C}{\pi} \Rightarrow C = \frac{\pi}{2}$$

$$P\{0 < X < 0.5\} = \int_0^{0.5} \frac{\pi}{2} \sin(\pi u) du = -\frac{1}{2} \cos \pi u \Big|_0^{0.5} = 0.5$$

$$P\{0 < X \leq 0.5\} = P\{0 < X < 0.5\} = 0.5, \quad (\text{since it is continuous})$$

$$P\{X = 0.6\} = 0, \quad (\text{since it is continuous})$$

(c) Valid

$$1 = \int_0^1 C \exp(-2u) du = -\frac{C}{2} \exp(-2u) \Big|_0^1 = \frac{C[1 - e^{-2}]}{2}$$

$$\Rightarrow C = \frac{2}{1 - e^{-2}}$$

$$P\{0 < X < 0.5\} = P\{0 < X \leq 0.5\} = \int_0^{0.5} \frac{2}{1 - e^{-2}} \exp(-2u) du$$

$$= \frac{1 - e^{-1}}{1 - e^{-2}} = \frac{1}{1 + e^{-1}}$$

$$P\{X = 0.6\} = 0, \quad \text{since it is continuous}$$

(d) Valid

$$1 = \int_{-1}^1 C(1-u) du = -C \frac{(1-u)^2}{2} \Big|_{-1}^1 = 2C \Rightarrow C = 0.5$$

$$P\{0 < X < 0.5\} = P\{0 < X \leq 0.5\} = \int_0^{0.5} 0.5(1-u) du =$$

$$= \frac{-1}{4} (1-u)^2 \Big|_0^{0.5} = -\frac{1}{4} \left(\frac{1}{4}\right) + \frac{1}{4} = \frac{3}{16}$$

$$P\{X = 0.6\} = 0, \text{ since it is continuous}$$

(e) Not valid, as it is negative in the range $1 < u < 2$

3-8. X is Gaussian, $m = 2$, $\sigma = 3$

$$f_X(u) = \frac{\exp\left\{-\left[\frac{(u-2)^2}{2 \bullet 9}\right]\right\}}{\sqrt{2\pi} \ 3} = \frac{\exp\{-(u-2)^2 / 18\}}{\sqrt{2\pi} \ 3}$$

Define $\Phi(x) = \int_{-\infty}^x \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du$ = unit cumulative Gaussian distribution.

$$P\{0 < X < 3\} = \Phi\left(\frac{3-m}{\sigma}\right) - \Phi\left(\frac{0-m}{\sigma}\right) = \Phi\left(\frac{1}{3}\right) - \Phi\left(\frac{-2}{3}\right) = 0.6305 - 0.2525 = 0.3780$$

$$P\{-1 < X < 3\} = \Phi\left(\frac{3-2}{3}\right) - \Phi\left(\frac{-1-2}{3}\right) = \Phi\left(\frac{1}{3}\right) - \Phi(-1) = 0.6305 - 0.1587 = 0.4718$$

$$P\{X < 5\} = \Phi\left(\frac{5-2}{3}\right) = \Phi(1) = 0.8413$$

$$P\{-1 < X\} = 1 - \Phi\left(\frac{-1-2}{3}\right) = 1 - \Phi(-1) = 1 - 0.1587 = 0.8413$$