

Additional Explanations for Problem 2-13

The problem is of a binary channel, where the conditional error probabilities are provided, as well as the probability of transmitting a “one” or a “zero”. We shall not discuss the case of the “undecided” bit at the receiver in this note.

We have the events:

$$T_0 = \{\text{a “zero” is sent}\}$$

$$T_1 = \{\text{a “one” is sent}\}$$

Clearly, the probabilities of T_0 and T_1 depend on the make-up of the message being sent, and we may call these the “a priori” probabilities of a bite being a “zero” or a “one”.

If we perform a test we may be able to find the conditional error probabilities at the receiver; namely, given what bit was sent, what is the probability that it was received incorrectly. We define for the received bits the following events:

$$R_0 = \{\text{a “zero” is received}\}$$

$$R_1 = \{\text{a “one” is received}\}$$

Let us also define the event that a bit is received in error:

$$A = \{\text{a bit is received in error}\}.$$

We therefore call the following probabilities, the conditional error probabilities:

$$P\{A|T_1\} = P\{R_0|T_1\} = P\{\text{a “zero” is received given that we know that a “one” is sent}\}$$

$$P\{A|T_0\} = P\{R_1|T_0\} = P\{\text{a “one” is received given that we know that a “zero” is sent}\}$$

Note that one way of interpreting the above probabilities, is to use relative frequency terms: the $P\{R_0|T_1\}$ describes the fractions of transmitted “ones” that are received as zeros, while $P\{R_1|T_0\}$ describes the fractions of transmitted “zeros” that are received as “ones”.

If we now wish to find the error probability in any received bit if we do **NOT** know what was sent, then we have to average over all possibilities, including the fractions of “ones” and “zeroes” sent. An error occurs if “one” is sent and a “zero” is received (we cannot use conditional probabilities here, since we do not know what kind of bit was sent!), or a “zero” is sent and a “one” is received:

$$\begin{aligned} A = \{\text{Error in a bit}\} &= \{(\text{a “zero” is sent but a “one” is received}) \\ &\quad \text{or (a “one” is sent but a “zero” is received)}\} = \\ &= \{(R_1 \cap T_0) \cup (R_0 \cap T_1)\} \end{aligned}$$

Hence, since the two events shown above are mutually exclusive, we have for the error probability:

$$\begin{aligned} P\{\text{error}\} &= P\{(R_1 \cap T_0) \cup (R_0 \cap T_1)\} = P\{R_1 \cap T_0\} + P\{R_0 \cap T_1\} = \\ &= P\{R_1 | T_0\}P\{T_0\} + P\{R_0 | T_1\}P\{T_1\} \end{aligned}$$

Note that if the conditional probabilities are the same, the error probability is the same as the conditional probability. However, suppose that we send 3 times as many “zeroes” as “ones”, we have to weigh the probability that a “zero” is received in error by the fractions of zeros that are transmitted, which is $3/4$, since $3/4$ of the transmitted bits are zeroes.

You may also consider the case of students in a class and the event is failing an exam. If we know the fraction of female students that fail, and male students that fail, we obtain the conditional failure based on the student gender. In order to obtain the probability that a student failed (without knowing the gender) we have to average over both gender, and this probability is affected by the fractions of males to females in the class.