

# Complementarities in Spectrum Markets

Hang Zhou\*, Randall A. Berry\*, Michael L. Honig\* and Rakesh Vohra†

\*EECS Department

Northwestern University, Evanston, IL 60208  
{hang.zhou, rberry, mh}@eecs.northwestern.edu

†CMS-EMS, Kellogg School of Management  
Northwestern University, Evanston, IL 60208  
r-vohra@kellogg.northwestern.edu

**Abstract**—It is widely recognized that current policies for allocating wireless spectrum have led to inefficiencies and under-utilization. One proposed solution to this is to enable “spectrum markets”, which allow for entities to sell and/or lease spectrum dynamically over time. In this paper we consider the design of such a market and how this is influenced by the underlying properties of the wireless medium. In particular, we focus on the role of interference created by different agents who may purchase the use of the same spectrum band at nearby locations. Such interference can result in “complementarities” among the spectrum goods being traded, which complicates the design of an efficient market mechanism. We give a simple model for such complementarities and for which the efficient allocation of spectrum assets to agents can be formulated as an integer program. We characterize the computational complexity of this problem and the performance of two different linear relaxations. We also comment on the optimal prices for such markets.

## I. INTRODUCTION

The continued growth of high-capacity wireless networks depends critically on the availability of adequate radio spectrum. It is widely recognized that current spectrum policy, which determines this availability, has several drawbacks, and is inhibiting this growth. The radio spectrum is a regulated resource, and in the U.S. has been partitioned among a variety of wireless applications by the FCC. As a result, much of the radio spectrum, which could be used to provide broadband wireless services, is often idle or under-utilized [1]. Various types of spectrum markets have been proposed [2]–[8] to facilitate trading of spectrum assets and enable efficient, flexible and dynamic allocation of radio spectrum resources. The design of such market involves both technical and economic issues. This is due primarily to the fact that transmitting in the same spectrum at nearby locations creates interference. This property of a wireless medium differentiates spectrum goods from other common goods. Namely, an entity’s value for a particular spectrum asset may, because of interference, depend on the use of other nearby assets. Allowing entities to purchase assets so as to preclude another agent from using them to create interference would mitigate this problem. This results in complementarities in

“bundles” of spectrum assets (i.e., the value of a bundle of assets may be greater than the sum of the values of each asset alone). Instead of not using an asset to prevent interference, an agent could also transmit utilizing some technique to mitigate the interference on a neighboring asset which it also owned, or even utilize transmitters in two adjacent assets for some form of “cooperative” transmission scheme. Such approaches would again lead to complementarities. Complementarities for spectrum assets may also exist across adjacent frequency bands (e.g., an agent may desire such bands to limit out-of-band interference and enable simpler receiver filters). In general, complementarities complicate the design of an efficient market mechanism.

In this paper, we study a simple model for such complementarities in a spectrum market, with a focus on how these effect the computation of an efficient allocation of spectrum and the resulting optimal prices. The efficient allocation of spectrum in the proposed model is formulated as an integer program, which in general is shown to be NP hard. We consider two different linear relaxations of the problem and study the optimal solutions of these relaxations. The first relaxation is shown to be not exact in even relatively small networks. The second relaxation is shown to be stronger. We also characterize classes of problems that are tractable. We then provide a numerical study of several heuristic algorithms for this problem and finally conclude with a discussion on optimal market prices.

## II. A MODEL FOR SPECTRUM ASSETS

Let  $C$  denote the set of spectrum assets available within a given geographic area. For example, each asset  $j \in C$  could represent the right to exclusively transmit with a fixed power mask over a given frequency band within a given geographic area.<sup>1</sup> Our interest here is on a scenario where  $|C|$  is large so that there many such assets available to be allocated, and these assets are small enough relative to the given power mask that interference effects among them are significant.<sup>2</sup>

<sup>1</sup>A detailed discussion on the definitions of spectrum assets and structures of spectrum market can be found in [9].

<sup>2</sup>Note this differs from most current spectrum auctions, in which the assets being allocated are large (e.g. consisting of several states) and thus in general interference effects between assets are typically minor.

Denote by  $A$  the set of agents who wish to acquire these assets. Pairs of interfering assets are represented via an *interference graph*, in which  $C$  is the set of nodes and the set of edges  $E$  corresponds to pairs of interfering assets (modeling interference in space or frequency). Let  $r_{ij}$  denote the revenue that agent  $i$  accrues when assigned asset  $j$  if there is no interference from any other asset  $j'$  such that  $(j, j') \in E$ . As an initial model for interference, we assume that if agent  $i$  is assigned asset  $j$  and agent  $q \neq i$  is assigned asset  $j'$ , where  $(j, j') \in E$ , then agent  $i$  suffers an interference cost of  $c_{jj'}^i$  and agent  $q$  suffers an interference cost of  $c_{jj'}^q$ .<sup>3</sup> On the other hand if agent  $i$  acquires both assets  $j$  and  $j'$ , she will not have an interference cost. For example, suppose that  $r_{ij} = 5$ ,  $r_{ij'} = 0$ , and  $c_{jj'}^i = 2$ . Then, if agent  $i$  acquires only band  $j$ , she will value it at  $5 - 2 = 3$ . If she acquires only band  $j'$ , she will value it at 0, but acquiring both  $j$  and  $j'$ , her value will be 5, which exceeds the sum of the values for  $j$  and  $j'$ , i.e. these bands are complements.<sup>4</sup>

### A. Efficient Allocation

From an economic perspective a common objective of any resource allocation is to maximize efficiency, meaning the total utility (revenue minus cost) derived from the allocation (e.g., summed over all agents requesting the resource). We consider finding such an allocation for the model introduced in the previous section. Let  $x_{ij} = 1$  if agent  $i \in A$  is assigned asset  $j \in C$  and zero otherwise. The efficient allocation of assets to agents is then given by the following integer program:

$$\begin{aligned} \max \quad & \sum_{i \in A} \sum_{j \in C} r_{ij} x_{ij} - \sum_{i \in A} \sum_{j, j' \in E} c_{jj'}^i (x_{ij} - x_{ij'})^+ \quad (1) \\ \text{s.t.} \quad & \sum_{i \in A} x_{ij} \leq 1, \quad \forall j \in C \\ & x_{ij} \in \{0, 1\}, \quad \forall i \in A, j \in C. \end{aligned}$$

The objective function of this problem is concave but non-differentiable. Note that if  $c_{jj'}^i = 0 \forall i \in A$  and  $(j, j') \in E$ , then Problem (1) becomes a simple allocation problem without any complementarities. In this case one should simply give each spectrum asset  $j$  to the agent with the largest value of  $r_{ij}$ . Our focus here is on cases where  $c_{jj'}^i > 0$ .

### III. AN INITIAL LINEAR RELAXATION

To begin we consider the following natural linear relaxation of (1):

$$\begin{aligned} \max \quad & \sum_{i \in A} \sum_{j \in C} r_{ij} x_{ij} - \sum_{i \in A} \sum_{(j, j') \in E} c_{jj'}^i d_{jj'}^i \quad (2) \\ \text{s.t.} \quad & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in C \\ & d_{jj'}^i \geq x_{ij} - x_{ij'} \quad \forall i \in A, (j, j') \in E \\ & d_{jj'}^i \geq 0 \quad \forall i \in A, (j, j') \in E, \end{aligned}$$

<sup>3</sup>Note that in general we allow these costs to be directional so that  $c_{jj'}^i$  need not be equal to  $c_{j'j}^i$ .

<sup>4</sup>Here we are assuming that the given spectrum is scarce enough so that if agent  $i$  does not acquire it, then another agent will.

$$0 \leq x_{ij} \leq 1 \quad \forall i \in A, j \in C$$

We next give two simple examples which show that even for small networks, this relaxation may not be exact, i.e., it may have fractional solutions. We will also see that fractional solutions can happen regardless of the amount of interference.

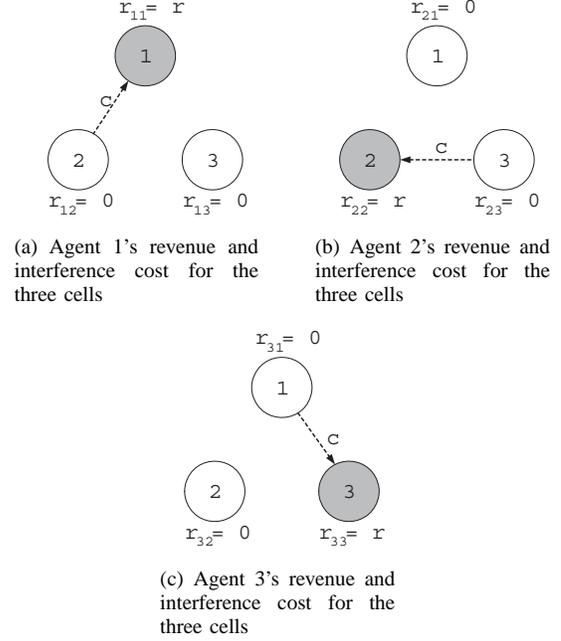


Fig. 1: A three agent/three asset example. Each shaded node denotes the asset each user has positive revenue for. The dashed arrows denote the interference a user would suffer without having the corresponding neighboring asset.

The first example we consider is a scenario with three agents ( $A = \{1, 2, 3\}$ ) and three spectrum assets ( $C = \{1, 2, 3\}$ ) as illustrated in Figure 1. Assume that  $r_{11} = r_{22} = r_{33} = r$  and  $r_{ij} = 0$  if  $i \neq j$ , i.e. each agent  $i$  only has positive revenue for asset  $j = i$ . Furthermore, assume that when each agent  $i$  is allocated asset  $i$ , she receives an interference cost of  $c$  if another agent is allocated asset  $i + 1 \pmod 3$ , i.e.  $c_{21}^1 = c_{32}^2 = c_{13}^3 = c$ . All other interference costs are zero. Consider the following fractional feasible solution to Problem (2): set  $x_{11} = x_{12} = x_{22} = x_{23} = x_{33} = x_{31} = 1/2$  and all of other  $x_{ij} = 0$ . This results in a total revenue of  $3r/2$ . Note that there is no interference cost incurred with this solution. On the other hand, the feasible integer solutions to this problem yields pay-offs of  $r$ ,  $3r - 3c$  and  $2r - c$ . Therefore, by comparing the total revenue, we conclude that if  $c > 1/2r$ , then the preceding fractional solution is better than any integral solution, i.e. the underlying LP's solution is fractional.

In the above example, the solution to the linear relaxation in (2) is integral for small enough interference (i.e. small values of  $c$ ), and fractional for large interference. In other words, whether the linear relaxation is exact depends on the amount of interference. We will next describe another example, in

which we show that whether the linear relaxation is exact does not necessarily depend on the level of interference.

In this example we consider a problem with four agents and four spectrum assets. The revenue and interference costs of each agent are illustrated in Figure 2. Each agent wants three consecutive assets and will receive interference if she does not have all three.

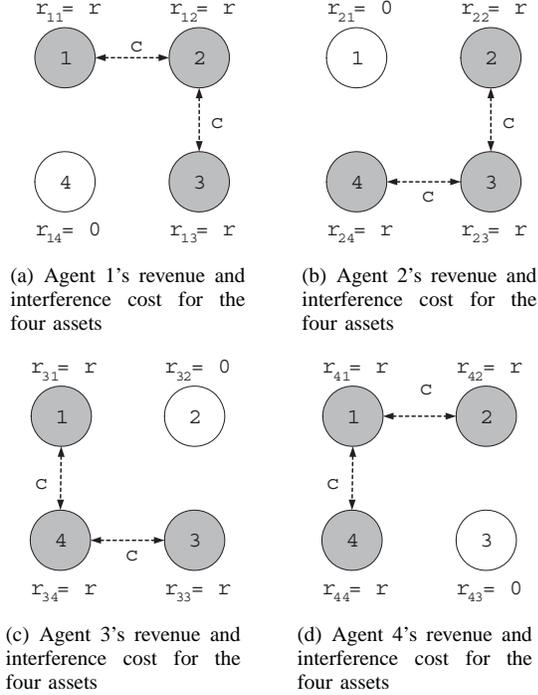


Fig. 2: An example with four agents and four assets. Each shaded asset denotes the asset each user has positive revenue for. Dashed arrows denotes the interference a user would suffer without having the corresponding neighboring asset.

It can be seen that there are exactly three agents having positive revenue for each asset, and dividing each asset evenly among the agents with positive revenue for the cell ( $1/3$  in this case) can always achieve a revenue of  $4r$ , which is the maximum revenue that can be achieved. But any integral solution achieving a total revenue of  $4r$  has to suffer some interference. As a result, the  $1/3$ -fractional solution is always superior to any integral solution (with  $c > 0$ ). Note that the above argument holds for any choice of interference costs (which need not be symmetric). Therefore, even for arbitrarily small interference, the relaxation can be inexact.

We also want to point out that fractional solutions cannot necessarily be interpreted as frequency sharing or time sharing. For example again consider the fractional solution for the example in Figure 1, where two agents equally share each asset. Consider implementing this solution using time sharing for example by dividing the total time that an asset is allocated into two equally sized time-slots. Next suppose that agent 1 is allocated asset 1 during the first time-slot. Then to not incur any interference costs (as in the fractional solution) agent 1 should also be allocated asset 2 during the

first time-slot. by the same reasoning, agent 2 would then be allocated assets 2 and 3 during the second time-slot (since agent 1 is already allocated asset 2 during the first time-slot). This would then require that agent 3 use slot 1 for asset 3, but slot 2 for asset 1, in which case she would incur an interference cost not accounted for in the solution to the LP. A similar argument can be made for the infeasibility of simple frequency sharing.

### A. Extreme Points of the Relaxation

In this section, our attention will be focused on the extreme points of the linear relaxation. We shall see that the number of fractional extreme points may grow exponentially with the size of the spectrum assets and the number of agents. It is also a suggestion of the difficulty of solving the original integer allocation problem.

Extreme points of the feasible region of the relaxation problem are characterized by the following constraints:

$$\sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in C \quad (3)$$

$$d_{jj'}^i \geq x_{ij} - x_{ij'} \quad \forall i \in A, (j, j') \in E \quad (4)$$

$$d_{jj'}^i \geq 0 \quad \forall i \in A, (j, j') \in E \quad (5)$$

$$0 \leq x_{ij} \leq 1 \quad \forall i \in A, j \in C \quad (6)$$

It takes a certain number of constraints to form an extreme point. We also, by convenient abuse of notation, denote the number of agents and the number of assets by  $A$  and  $C$ . Without any assumption of the topology, we consider all possible edges between the nodes, i.e., the interference graph is fully connected. So we have  $AC$   $x$  variables and  $C(C-1)$   $d$  variables. Therefore, we need  $AC + C(C-1)$  independent binding constraints to form an extreme point. Note that any feasible integral solution gives us exactly this number of binding constraints. Obviously, any feasible integral solution is an extreme point.

We next consider general cases including fractional solutions. Since for any  $d_{jj'}^i$  we can always have either (4) or (5) binding, we only need  $AC$  independent binding constraints for  $x_{ij}$ . An observation is that if  $x_{ij} = 1$ , it also suggests that  $x_{i'j} = 0 \quad \forall i' \neq i$ , which offers  $A$  constraints in total. So we can ignore this asset and focus on the other  $C-1$  assets as if these were the only assets. Thus, without loss of generalities, we assume  $0 \leq x_{ij} < 1 \quad \forall i \in A, j \in C$ . Note that (5) implies  $x_{ij} = x_{ij'}$  or  $d_{jj'}^i = 0$ . Let  $S_f^i = \{j : 0 < x_{ij} < 1\}$  be the set of assets which agent  $i$  is assigned fractions. Define a *group* of assets of agent  $i$ ,  $c_i = \{j : x_{ij} = \epsilon_i\}$  for some  $0 < \epsilon_i < 1$ . In other words, a group is the set of assets which an agent is assigned the same fractional  $x_{ij}$ . Suppose user  $i$  has  $k_i$  groups, then it can be seen that this provides  $C - k_i$  constraints. Note that a singleton group does not provide any binding constraint. So summing over all agents' groups, we have  $\sum_{i \in A} (C - k_i) = AC - \sum_{i \in A} k_i$  binding constraints. Together with  $C$  constraints given in (3), we have  $AC - \sum_{i \in A} k_i + C$  constraints. In order to form an extreme point, we need  $AC - \sum_{i \in A} k_i + C \geq AC$ , i.e.

$$\sum_{i \in A} k_i \leq C. \quad (7)$$

The inequality in (III-A) gives the condition for an extreme point. In other words, the total number of groups can never exceed  $C$ . This can also be understood from the view of solving  $C$  linear equations. Because for each group, there is a variable associated with it. If it is an extreme point, then we must be able to solve these variables from the linear equations given in (3). Since there are at most  $C$  linear independent equations, we can solve for at most  $C$  variables, which correspond to  $C$  groups.

As a special case, if  $A = C$   $k_i > 0 \forall i$ , then we have  $k_i = 1 \forall i$ . Namely, if the number of agents and the number of assets are the same, and each agent has some fractional  $x$ 's, then each agent's fractional  $x$  is unique. However, for general  $A$  and  $C$ , there could be many ways of selecting  $k_i$  to meet the condition in (III-A). As a result, there could be many fractional extreme points as the number of agents and number of assets grow. This suggests that solving the integer allocation problem may be complex; we show that this is indeed the case in the following section.

#### IV. COMPUTATIONAL COMPLEXITY

In this section we consider the computational complexity of Problem (1) and show that in general the problem is NP hard. We show this in two different cases. First for a general version of the problem and then for a version where the interference cost terms are required to be small.

*Proposition 1:* The spectrum asset allocation problem in (1) is NP-hard.

*Proof:* We prove this by showing that any instance of the *maximum independent set* problem can be transformed into an instance of this problem. Suppose we want to find a maximum independent set in a given graph  $G = (V, E)$ . We then construct an instance of the spectrum asset assignment problem in which  $G$  is the interference graph and  $N$  represents the set of spectrum assets. Let the number of agents be equal to the number of nodes in  $G$ . Set  $r_{ij} = 1$  only if the index of an agent and a cell agree, i.e.,  $i = j$ ; otherwise  $r_{ij} = 0$ . Set  $c_{jj'}^i = 2 \forall i \in A$  and  $(j, j') \in E$ . With these values, it can be seen that in a optimal solution to (1) if agent  $i$  is allocated spectrum asset  $i$ , then no other agents will be allocated any neighboring asset to  $i$ . Thus a solution to (1) must correspond to a maximum independent set in  $G$ . Since the maximum independent set problem is NP-complete, the result follows. ■

The previous proof required that the interference cost of an agent between two neighboring assets be larger than the revenue achieved by that agent from owning both of the assets. In most spectrum markets, such a high interference costs may seem unreasonable and one might hope that with small enough interference that the complexity improves. However, as the next proposition shows, even with arbitrarily small interference costs, this problem can still be NP-hard.

*Proposition 2:* The spectrum asset allocation problem in (1) is NP-hard even if the interference costs on each link is arbitrarily small (relative to the revenue).

*Proof:* For this proof, we will show that any instance of the *Graph Partitioning* problem can be transformed (in

polynomial time) into an instance of the spectrum asset allocation problem with arbitrarily small interference costs. Given a graph  $G = (V, E)$  and positive number  $K \geq 3$ , the *Graph Partitioning* problem is to find a partition of  $V$  in to disjoint sets  $V_1, \dots, V_m$  such that  $|V_i| \leq K$  for all  $1 \leq i \leq m$  and such that if  $E' \subset E$  is the set of edges that have two endpoints in two different sets, then  $|E'|$  is minimized, where  $|\cdot|$  is the cardinality of the set. (see [10] for a general version of this problem). This problem is NP-complete, even with the restriction that  $K = 3$

We now give a transformation of the graph partitioning problem with  $K = 3$  into the spectrum asset allocation problem. Let  $V$  be the set of spectrum assets and  $G$  the corresponding interference graph. For any  $V_i \subset V$  such that  $|V_i| \leq 3$ , introduce an agent with  $r_{ij} = r_0$  only for  $j \in V_i$ , and zero otherwise; also, set  $c_{jj'}^i = c_0$  for all edges  $(j, j') \in E$  such that  $j \in V_i$  or  $j' \in V_i$  or both. The number of agents resulting from this  $\frac{1}{6}n^3 + O(n)$ . Thus, this transformation can be done in polynomial time. Furthermore, assume  $r_0 > 0$  and  $c_0 > 0$  are chosen such that  $c_0$  is small enough so that an agent's revenue is always greater than the costs she suffers, as well as the costs she imposes on the agents owning the neighboring assets. This can always be done and allows for  $c_0$  to be arbitrarily small relative to  $r_0$ . As a result of this choice, each asset  $j$  will be allocated to some agent with for which  $r_{ij} = r_0$ . Hence, the first summation term representing the total revenue across all assets in the objective function in (1) becomes a constant ( $= |V|r_0$ ) and the optimal solution is that the one which minimizes the total interference costs among the assets. Since all interference costs are assumed to be equal, the total interference cost of an assignment is the number of edges whose two end nodes belong to different agents. By construction, the number of assets assigned to a given agent will be no greater than 3 and hence the solution to this also a solution to the graph partitioning problem. ■

#### V. COMPUTATIONALLY TRACTABLE SCENARIOS

From the discussion in Sections III and IV, it is shown that Problem (1) may be computationally intractable. However, we are able to identify two scenarios in which the integer allocation problem can be solved in polynomial time. In the first scenario, interference is small enough to be ignored. The second scenario is for a special case of the interference graph, namely when this graph is a line.

The first scenario we consider is to give conditions on the interference costs and revenues so that a simple greedy assignment of each asset to the agent who values it the most is optimal. This is true in either of the following cases:

**Case 1:** If for each asset, there is only one agent has positive revenue, and the revenue satisfies

$$r_{ij} \geq \sum_{j':(j,j') \in E} c_{jj'}^i + \sum_{j':(j,j') \in E} \max_{i \in A} c_{j'j}^i, \quad (8)$$

then the solution is integral, i.e.,  $x_{ij} = 1$  and  $x_{ij'} = 0 \forall j' \neq j$ .

**Case 2:** This is a generalization of Case 1. Suppose each

agent only has positive revenue for one asset, but there can be many agents with positive revenue for the same asset. If there is at least one agent's revenue satisfying condition in (8), then the solution is integral with  $x_{ij^*} = 1$  where  $j^* = \operatorname{argmax}_j (r_{ij} - \sum_{j':(j,j') \in E} c_{jj'}^i)$ . The result in Case 1 suggests that a asset should be allocated to a agent with positive revenue. As a result, the agent who has asset  $j$  in the optimal solution will not have any neighboring assets by the assumption. Therefore, among the agents with positive revenue for asset  $j$ , the asset should be allocated to the agent with the largest payoff (revenue minus costs).

In the second scenario (the line model), we model the problem in a different way in order to see the tractability. We use the term "interval" to denote consecutive assets. Let  $\mathcal{I}$  be the set of all intervals on the line (of all possible lengths), and  $u_i(I)$  be the utility of agent  $i$  having interval  $I$ . Here  $u_i(I)$  denotes the revenue that agent  $i$  gets if she does not own the left or right neighboring assets, i.e.  $u_i(I) = \sum_{j \in I} r_{ij} - c_{j_1(j_1-1)}^i - c_{j_2(j_2+1)}^i$ , where  $I = [j_1, j_2]$ . Let  $x_{iI} = 1$  if interval  $I \in \mathcal{I}$  is assigned to agent  $i \in A$ . Then, the problem can be written as

$$\max \sum_{i \in A} \sum_{I \in \mathcal{I}} u_i(I) x_{iI} \quad (9)$$

subject to

$$\sum_{i \in A} \sum_{I \ni j} x_{iI} \leq 1 \quad \forall i \in A, j \in C, I \in \mathcal{I}$$

$$x_{iI} \in \{0, 1\} \quad \forall i \in A, I \in \mathcal{I}$$

*Lemma 1:* The linear system defined by

$$\sum_{i \in A} \sum_{I \ni j} x_{iI} \leq 1 \quad \forall i \in A, j \in C, I \in \mathcal{I}$$

$$0 \leq x_{iI} \leq 1, \quad \forall i \in A, I \in \mathcal{I}$$

has only integral extreme points.

This lemma holds simply because the constraint matrix in (9) has the property of consecutive 1's in each column. As a result, the constraint matrix is *totally unimodular* [11].

Because of Lemma 1, the problem in (9) can be solved by solving its linear relaxation. Namely, it can be solved in polynomial time. Last, since the way in which  $u_i(\cdot)$  is defined, the solution to problem in (9) must be a solution to the allocation problem in (1). Therefore, the integer allocation problem in (1) can be solved in polynomial time in this case.

## VI. AN ALTERNATIVE FORMULATION

In this section we consider a reformulation of 1 that follows from replacing the  $(x_{ij} - x_{ij'})^+$  terms in the objective with  $x_{ij}(1 - x_{ij'})$ . This yields the following equivalent

problem:

$$\max \sum_{i \in A} \sum_{j \in C} (r_{ij} - \sum_{jj' \in E} c_{jj'}^i) x_{ij} + \sum_{i \in A} \sum_{jj' \in E} c_{jj'}^i x_{ij} x_{ij'} \quad (10)$$

$$\begin{aligned} \text{s.t. } & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in C \\ & x_{ij} \in \{0, 1\} \quad \forall i \in A, j \in C. \end{aligned}$$

For each  $i \in A$  and  $j \in C$ , let  $\tilde{r}_{ij} = (r_{ij} - \sum_{jj' \in E} c_{jj'}^i)$ ; these terms term in the objective can be interpreted as the minimum revenue that agent  $i$  can gain from asset  $j$  (assuming that she receives interference from all neighboring assets). The term  $c_{jj'}^i x_{ij} x_{ij'}$  can be viewed as the extra revenue gained if agent  $i$  receives two complementary assets  $j$  and  $j'$ .

This yields the following linear relaxation:

$$\max \sum_{i \in A} \sum_{j \in C} \tilde{r}_{ij} x_{ij} \sum_{i \in A} \sum_{jj' \in E} c_{jj'}^i z_{jj'}^i \quad (11)$$

$$\begin{aligned} \text{s.t. } & \sum_{i \in A} x_{ij} \leq 1, \quad \forall j \in C \\ & z_{jj'}^i \leq x_{ij}, \quad \forall i \in A, (j, j') \in E \\ & z_{jj'}^i \leq x_{ij'}, \quad \forall i \in A, (j, j') \in E \\ & 0 \leq x_{ij} \leq 1, \quad \forall i \in A, j \in C. \end{aligned} \quad (12)$$

The next lemma shows that this relaxation is equivalent to the relaxation in (2) in terms of the solutions it generates.

*Lemma 2:* The relaxation in (2) and (11) have the same optimal solution and the same set of optimal  $\{x_{ij}\}$  values.

*Proof:* Let  $\{x_{ij}, z_{jj'}^i\}$  be an optimal solution to Problem (11), with an objective value of  $f^*$ . Note that at any optimal solution to Problem 11, it must be that

$$z_{jj'}^i = \min(x_{ij}, x_{ij'}). \quad (13)$$

Consider assigning the same  $x_{ij}$  values to the variables in Problem 2 and set

$$d_{jj'}^i = x_{ij} - z_{jj'}^i$$

for all  $i \in A$  and  $jj' \in E$ . Using (13), it follows that this choice of variables is feasible for Problem 2. Furthermore, by construction it results in an objective value of  $f^*$  for Problem 2, i.e. the optimal solution to Problem 2 is no less than the optimal solution to Problem 11.

To complete the proof,  $\{x_{ij}, d_{jj'}^i\}$  be an optimal solution to Problem 2 with an objective value of  $g^*$ . We can transfer this into a feasible solution to Problem 11 with an objective value no less than  $g^*$  using the same  $x_{ij}$  values and setting  $z_{jj'}^i = x_{ij} - d_{jj'}^i$  (noting that at any optimal solution to Problem 11 it must be that  $d_{jj'}^i = (x_{ij} - x_{ij'})^+$ ). ■

So far this formulation does not appear to have any advantages compared to the previous one. However, we will next modify it by adding additional constraints, which will result in a stronger formulation. To begin, consider Problem (10) for the example with three spectrum assets shown in Figure 1. In this example, it can be seen that the  $z_{jj'}^i$  variables can be non-zero for at most one agent  $i$ , since if one agent

has  $z_{jj'}^i = 1$  it must have both the  $j$  and  $j'$  spectrum assets and so no other agent can have two assets. Based on this observation, we can then add the following constraints to Problem (11)

$$z_{jj'}^i + \sum_{i' \neq i} z_{kj}^{i'} + \sum_{i' \neq i} z_{j'k}^{i'} \leq 1, \quad \forall jj' \in E, i \in A, \quad (14)$$

where  $k \in C$  is the asset not included in the edge  $jj'$ . Doing this gives a stronger linear relaxation of Problem (10) for this three node example. In particular, this additional constraint eliminates the fractional optimal solution in the example for  $c > 1/2r$  since achieving this solution would require that  $z_{21}^1 = z_{32}^2 = z_{13}^3 = 1/2$  and so violate (14). Indeed, it can be shown that with these additional constraints the resulting linear relaxation for any problem with three spectrum assets will be exact.

Given an arbitrary interference graph  $G$ , we can add such constraints as in (14) for any subgraph of  $G$  which is also a triangle. Next, we generalize this type of constraint to a square graph as in Figure 2. For such a graph it can be seen that the  $z_{jj'}^i$  variable can be non-zero for at most two agents, and furthermore if one agent has more than one non-zero  $z_{jj'}^i$ , then it must be the only agent for which these variables are non-zero. This can be encoded using the following constraints

$$z_{jj'}^i + z_{kk'}^i + \sum_{i' \neq i} z_{ll'}^{i'} + \sum_{i' \neq i} z_{mm'}^{i'} \leq 2, \quad \forall i \in A, \quad (15)$$

where  $jj', kk', ll'$  and  $mm'$  represent some permutation of the edges in  $E$  (one set of constraints is needed for each of the six permutations in which  $jj'$  and  $kk'$  are not the same). Again, it can be seen that adding such constraints eliminates the fractional solution in the example. In an arbitrary interference graph  $G$ , such constraints can be added for any subgraph which is a square. Continuing in a similar manner we could develop similar constraints for larger cycles.

## VII. RING TOPOLOGIES

In this section we focus on a case where the interference graph is a ring (this includes the examples in Figures 1 and 2 as special cases. For such a graph, we consider the relaxation in (11). This relaxation may not be integral, however we will show in the following that it is “nearly” integral and that we can efficiently find a solution to the original integer program.

Number the nodes of  $G$  consecutively  $1, 2, 3, \dots, C$ . Then, the only edges in  $E$  are of the form  $(u, u+1 \bmod C)$ . In this case, the constraints for (11) become (not including the constraints that each variable be in  $[0, 1]$ ):

$$\sum_{i \in A} x_{iu} \leq 1, \quad \forall u \in C \quad (G_u)$$

$$z_{u,u+1}^i \leq x_{iu} \quad (R_{iu})$$

$$z_{u-1,u}^i \leq x_{iu} \quad (L_{iu})$$

*Lemma 3:* Consider removing one of the assets, that is pick a asset  $j \in C$  and an agent  $i \in A$  and set  $x_{ij} = 1$  and  $x_{kj} = 0$  for all  $k \neq j$ . The resulting constraint matrix given by  $(G_u)$ ,  $(R_{iu})$ , and  $(L_{iu})$  for  $u \in C$  is totally unimodular.

*Proof:* To do this, we will use the Ghouila-Houri characterization of totally unimodular matrices, which states that for each subset of the constraints,  $H$ , we need to find an appropriate partition of the constraints into two sets  $R$  (red) and  $B$  (blue) such that for all columns  $j$ , we have that  $|\sum_{i \in B} a_{ij} - \sum_{i \in R} a_{ij}| = 0, 1$ .

In particular this would mean

- 1)  $R_{iu}$  and  $L_{i,u+1}$  (if they occur in  $H$ ) must receive different colors.
- 2) If  $G_u, L_{iu}, R_{iu} \in H$ , then they can all receive the same color or  $G_u, L_{iu}$  or  $G_u, R_{iu}$  can receive the same color.
- 3) If only  $L_{iu}$  and  $R_{iu}$  occur in  $H$  (But not  $G_u$ ), they must receive different colors.
- 4) If only  $G_u$  and  $L_{iu}$  ( $R_{iu}$ ) are in  $H$ , they must receive the same color.

To accomplish this, mark each  $u \in C$  such that (i)  $G_u \in H$ , and (ii) there is an  $i$  such that either  $R_{iu} \in H$  or  $L_{iu} \in H$  but not both. Label each  $u$  such that  $G_u \notin H$ .

Pick the first consecutive string of  $u$ 's such that  $G_u \in H$  and color each  $G_u$  blue and red alternating. Suppose the last  $u$  in this string is colored Red. This string will terminate in a string of labeled  $u$ 's. Then we pick up another string of  $u$ 's for which  $G_u \in H$ . If the preceding string of labeled  $u$ 's is of even length then color the first  $u$  of the next string (with  $G_u \in H$ ) Red, otherwise color it Blue.

For each marked  $u$ , color  $R_{iu}$  ( $L_{iu}$ ) with the same color as the corresponding  $G_u$ . Observe that if  $u$  and  $u+1$  are marked, and  $R_{iu}$  and  $L_{i,u+1}$  are in  $H$ , they receive different colors.

If  $u$  is not marked but  $G_u \in H$ , then  $L_{iu}$  and  $R_{iu}$  are in  $H$ . Color  $L_{iu}$  and  $R_{iu}$  with the same color as  $G_u$ . Observe that if  $u$  or  $u-1$  are marked, then  $R_{iu}$  and  $L_{i,u+1}$  (as well as the other possible pairs) receive different colors.

Consider now a  $u$  such that  $G_u$  is not in  $H$ . Start with one such that  $G_{u-1}$  is in  $H$ . Color  $L_{iu}$  with a color opposite to that of  $R_{i,u-1}$ . Color  $R_{iu}$  with the same color as  $R_{i,u-1}$ . By the parity condition, the colors line up at the start and end of the string. ■

The above property immediately gives a polynomial algorithm for solving the original integer program.

## VIII. NUMERICAL RESULTS

In this section, we present some numerical results which compare the total utility achieved by several natural approximation schemes.

We consider a finite  $L_1 \times L_2$  grid of spectrum assets. Namely, the underlying interference graph is a grid as in Figure 3. We assume the interference cost is proportional to the corresponding revenue of an agent for a given asset, i.e.,  $c_{jj'}^i = \lambda r_{ij}$  if  $j'$  is a neighbor of  $j$ 's, where  $0 \leq \lambda \leq 1$  is a constant factor. Also, we assume that an agent will have a positive revenue for a cell with probability  $p$ , and zero with probability  $1-p$ . Note that smaller  $p$  will result in more interference. Furthermore, the agents' revenue is symmetric, i.e.,  $r_{ij} = r$  if  $r_{ij} > 0$ ,  $\forall i \in A$  and  $j \in C$ , where  $r$  is a positive constant. We consider the following approximate allocation schemes:

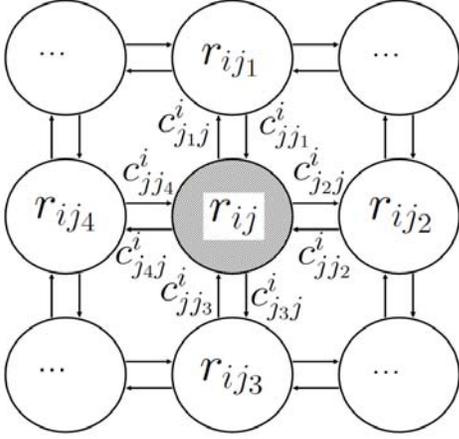


Fig. 3: Interference cost model with grid topology

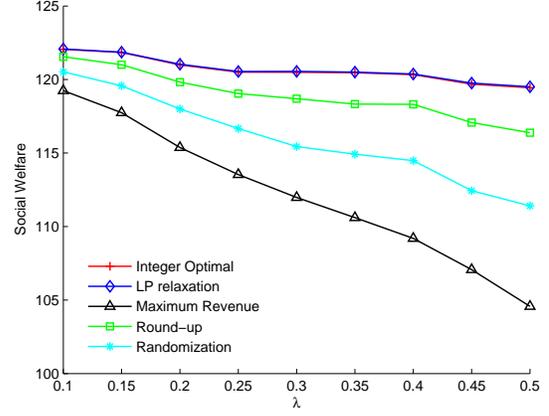
**Approximation 1:** maximum revenue scheme. In the maximum revenue scheme, a cell is assigned to the agent who accrues most revenue from it (ignoring any interference costs).

**Approximation 2:** round-up scheme. In the Round-up scheme, assets are allocated based the solution to the linear relaxation in (2). If the solution is fractional, then a asset is assigned to the agent with the largest fraction.

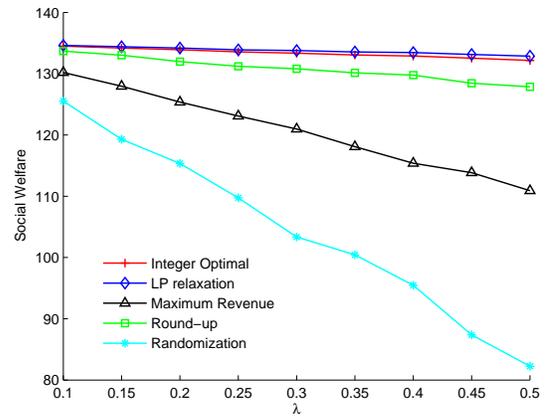
**Approximation 3:** randomization scheme. In the randomization scheme, the allocation is done based on the solution to the linear relaxation in (2). If the solution is fractional, then an asset is assigned to a agent with a probability equal to the agent's fraction.

In Figure 4, we show results for a system with  $L_1 = L_2 = 3$ ,  $A = 6$  and  $r = 15$ . The total utility is plotted against  $\lambda$  for different allocation schemes with the probability  $p = 0.3$  and  $p = 0.8$  respectively. Based on the numerical results, it can be seen that the social welfare of the integral solution and fractional solution are very close. In other words, the linear relaxation gives a very close upper bound on the social welfare of the spectrum market. Moreover, the round-up scheme performs well most of the time. However, the maximum revenue and randomization schemes suffer performance degradation with large interference (large  $\lambda$ ).

The linear relaxation and round-up scheme give upper and lower bound on the social welfare that the integer allocation can achieve. Thus, we show how the social welfare gap between the two changes with different sizes of network. In Figure 5, the social welfare gap between the LP relaxation and the round-up scheme is plotted as a percentage with a fixed number of agent, but different network sizes. The results in the figure show that for small  $p$ , the gap is smaller in larger networks. Furthermore, the gap increases with the amount of interference (the value of  $\lambda$ ).



(a)  $p = 0.3$



(b)  $p = 0.8$

Fig. 4: Social Welfare of various allocation schemes for a  $3 \times 3$  lattice with six agents. (a)  $p = 0.3$  (b)  $p = 0.8$

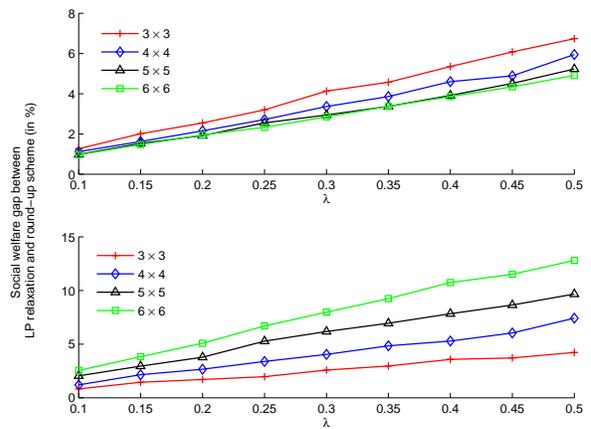


Fig. 5: Social welfare gap between LP relaxation and round-up scheme with different size of grids with  $A = 9$  and  $r = 15$ . Top:  $p = 0.3$ ; Bottom:  $p=0.8$ .

## IX. OPTIMAL MARKET PRICES

For a market model, the optimal market prices clear the market, and achieve the optimal social welfare. The analysis of the efficient allocation of the spectrum market can provide suggestions for the optimal market prices. If an integer problem can be solved through its linear relaxation, then advantage of the exact linear programming formulation is that the dual variables of the problem have an interpretation as prices. Thus, a primal-dual algorithm can suggest how prices should be adjusted to achieve the efficient allocation.

In Section III, it is shown that the linear relaxation in (2) is not always exact. This suggests that for this relaxation, there does not exist uniform per/cell prices which achieve the optimal total utility. In other words, if the optimal utility is the objective of the market, then the market has to assign prices for bundles of spectrum assets, and price differentiation among agents may be necessary as well. Therefore, the complementarities not only make the efficient market allocation difficult to compute, but also complicate the design of optimal pricing schemes.

Nevertheless, in the scenarios where the round-up scheme in Section VIII performs well (such as (a) in Figure 4, we can still obtain a reasonably good approximation by using the prices from the primal-dual iterations of the linear relaxation.

## X. CONCLUSIONS

We have studied a simple stylized model for complementarities that may arise in spectrum markets. We have focused mainly on determining an efficient assignment of spectrum assets to agents. We have shown that in general due to complementarities, this can be a computationally challenging problem, however in several special cases it can be solved efficiently. How relevant these computational issues will be for a spectrum market will of course depend on the number of spectrum assets available and the time-scale at which the market operates. Here we have focused on simply determining an efficient allocation given complete knowledge of the agents revenues and costs. A possible direction for future work is to consider a mechanism for acquiring this information from potentially strategic agents. Another open direction is to develop more refined models for interference costs and study how various definitions of spectrum assets influence these costs.

## REFERENCES

- [1] M. A. McHenry and D. McCloskey, "Spectrum occupancy measurements, Chicago, Illinois, November 16-18, 2005," Shared Spectrum Company Report, available at <http://www.sharespectrum.com/measurements/recent.html>, 2005.
- [2] M. M. Bykowsky, M. A. Olson, and W. W. Sharkey, "A market-based approach to establishing licensing rules: Licensed versus unlicensed use of spectrum," *FCC OSP Working Paper Series No.43*, February 2008.
- [3] J. Chapin and W. Lehr, "The path to market success for dynamic spectrum access technology," *Communications Magazine, IEEE*, vol. 45, no. 5, pp. 96–103, 2007.
- [4] —, "Time-limited leases in radio systems [Topics in Radio Communications]," *IEEE Communications Magazine*, vol. 45, no. 6, pp. 76–82, 2007.
- [5] A. S. de Vany, R. D. Eckert, C. J. Meyers, D. J. O'Hara, and R. C. Scott, "A property system for market allocation of the electromagnetic spectrum: A legal-economic-engineering study," *Stanford Law Review*, vol. 21, no. 6, pp. 1499–1561, June 1969.
- [6] L. Doyle and T. Forde, "Towards a fluid spectrum market for exclusive usage rights," *2nd IEEE International Symposium on New Frontiers in Dynamic Spectrum Access Networks (DySPAN 2007)*, pp. 620–632, 17-20 April 2007.
- [7] FCC, "Second report and order: Promoting efficient use of spectrum through elimination of barriers to the development of secondary markets," Federal Communications Commission Report 04-167, Sept. 2004.
- [8] J. M. Peha and S. Panichpapiboon, "Real-time secondary markets for spectrum," *Telecommunications Policy*, vol. 28, pp. 603–618, 2004.
- [9] J. Bae, E. Beigman, R. Berry, M. Honig, H. Shen, R. Vohra, and H. Zhou, "Spectrum markets for wireless services," *3rd IEEE Symposium on New Frontiers in Dynamic Spectrum Access Networks (DySPAN 2008)*, pp. 1–10, Oct. 2008.
- [10] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
- [11] G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization*. Wiley-Interscience, November, 1999.