Performance of Reduced-Rank Interference Suppression: Reflections and Open Problems

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In this article we describe some of the background leading up to our paper on reduced-rank interference suppression [1], along with some related unsolved problems. This work has its origins in much of the work on multiuser detection and interference suppression for Code-Division Multiple Access systems, which had been in progress for more than a decade. At the time this work was in progress, subspace, or reduced-rank methods for interference suppression were being considered by a few different authors (e.g., [2–7]) as a way to reduce complexity, improve robustness, and reduce estimation error in an adaptive mode.

This paper was inspired by two prior important developments in the design and analysis of linear filters for interference suppression: (i) the introduction of the Multi-Stage Wiener Filter (MSWF) [8, 9], and (ii) the large system analysis of CDMA introduced in [10–12]. The first author was fortunate enough to listen to a conference presentation on the MSWF during the Milcom '97 conference [9]. Subsequently, work began on the application of this technique to CDMA. Initial simulation results demonstrated that the subspace dimension needed by the MSWF to achieve the full-rank, or optimum (Minimum Mean Squared Error (MMSE)) performance was far below that required by other techniques being considered at the time, such as those which attempt to separate the signal and noise subspaces (e.g., Principle Components) [13, 14]. This served as the motivation for the subsequent large system analysis in [1].

Multi-Stage Wiener Filter

For the communications scenario considered, the classical Wiener filtering problem is to estimate a transmitted symbol b_1, given a noisy $N \times 1$ received vector

$$\mathbf{r} = \mathbf{S}\mathbf{b} + \mathbf{n} \tag{1}$$

where $\mathbf{b} = [b_1, \cdots, b_K]'$ is the $K \times 1$ vector of unit-variance transmitted symbols, $\mathbf{S}$ is an $N \times K$ signature matrix, and $\mathbf{n}$ is the noise vector with covariance matrix $\sigma^2\mathbf{I}$. The optimal (full-rank) linear filter is given by the vector $\mathbf{c}$, which minimizes the Mean Squared Error $E[\|b_1 - \mathbf{c}^\dagger\mathbf{r}\|^2]$

A block diagram of a MSWF is shown in Figure 1. The filter consists of the constituent "sub-filters" $\mathbf{c}_1, \cdots, \mathbf{c}_D$, the weights $w_1, \cdots, w_D$, and the blocking matrices

$\mathbf{B}_1, \cdots, \mathbf{B}_{D-1}$. (In the figure $D = 4$.) Referring to Figure 1, the blocking matrices satisfy $\mathbf{B}_n^\dagger \mathbf{c}_n = 0$, for each $n = 1, \cdots, D-1$. The subfilter $\mathbf{c}_n$ has input $\mathbf{r}_{n-1}$ and output d_n (i.e., $d_n = \mathbf{c}_n^\dagger \mathbf{r}_{n-1}$). Furthermore, $\mathbf{c}_n = E[d_{n-1}^* \mathbf{r}_{n-1}]$ (possibly normalized so that $\|\mathbf{c}_n\| = 1$), and can be interpreted as a matched filter for estimating d_{n-1} from $\mathbf{r}_{n-1}$.

The MSWF essentially decomposes the original estimation problem, i.e., estimate $d_0 = b_1$ from $\mathbf{r}_0 = \mathbf{r}$, into a sequence of subproblems, i.e., estimate d_n from $\mathbf{r}_n$ for $n = 1, \cdots, D-1$. At any stage n, if $\mathbf{c}_n$ is optimal (i.e., minimizes $E[|d_{n-1} - \mathbf{c}_n^\dagger \mathbf{r}_{n-1}|^2]$), then the associated truncated filter at stage n is equivalent to the full-rank MMSE filter. For example, if $\mathbf{c}_2$ is replaced by the associated MMSE filter, then the truncated filter consisting of $\mathbf{c}_1$, $\mathbf{B}_1$, $\mathbf{c}_2$, w_1 and w_2 is the classical generalized sidelobe canceller [15]. The MSWF is obtained by recursively expanding each subfilter according to the generalized sidelobe canceller structure. Truncating the filter to D stages corresponds to projecting the full-rank solution onto a D-dimensional subspace, and is a *reduced-rank* MSWF.

The reduced-rank MSWF has an appealing, regular structure, which intrigued the authors. Furthermore, it has relatively low complexity. Namely, adaptive versions of the MSWF require less computation than other reduced-rank techniques, which require an eigen-decomposition of the input covariance matrix [13,14]. This motivated the desire to obtain a deeper understanding of its structure and performance.

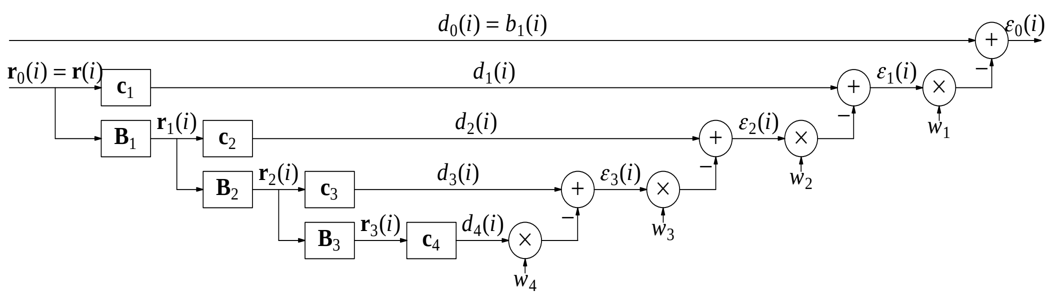

Figure 1: Multi-Stage Wiener Filter.

Large System Performance

In applications such as CDMA and multi-antenna systems, the mixing matrix $\mathbf{S}$ in the model (2) is typically random, which complicates the performance evaluation. Namely, it is generally difficult to evaluate the output Signal-to-Interference-Plus-Noise Ratio (SINR) averaged over $\mathbf{S}$. Interestingly, if the elements of $\mathbf{S}$ are *i.i.d.*, then a closed-form expression for the SINR can be obtained by letting K and N tend to infinity with fixed ratio $\alpha = K/N$.

The authors first became aware of this type of large system analysis for CDMA through the conference presentation [16]. This has proven to be a very powerful technique, and has been used by numerous authors in recent years to analyze the performance of various receivers with different types of Multi-Input/Multi-Output channels. For the linear estimation problem considered, Tse and Hanly [12] showed that the output SINR for the full-rank linear receiver (β) satisfies the fixed-point equation

$$\beta = \frac{1}{\sigma^2 + \alpha \frac{1}{1+\beta}} \tag{2}$$

(Here we assume that each element of $\mathbf{S}$ has mean zero and variance $1/N$.)

We began our large system analysis of the MSWF rank by rank. Namely, if the rank $D = 1$, the MSWF is simply a matched filter with large system SINR $\beta_1 = 1/(\alpha + \sigma^2)$. We obtained expressions for the SINR corresponding to $D = 2$ and $D = 3$ as well, but going beyond this seemed exceedingly tedious and messy. The expression in the paper was obtained by a good guess. Namely, we observed that the Tse-Hanly formula can be written as the continued fraction

$$\beta = 1/(\sigma^2 + \alpha/(1 + 1/(\sigma^2 + \alpha/(1 + ...)))) \tag{3}$$

Because of the regular, iterative structure of the MSWF, we suspected that the SINR as a function of rank D might be obtained by simply truncating this continued fraction. It was easy to verify numerically that this is indeed true. Proving it, however, was more challenging than we expected, and was accomplished over a period of a few months.

It occured to us after we guessed the expression for reduced-rank SINR that essentially full-rank performance can be obtained with a finite-rank MSWF, independent of the system size (K and N). That is, the preceding continued-fraction converges in about eight iterations, corresponding to $D = 8$, no matter how large K and N are. This came as a big surprise, since all other subspace techniques we knew of, such as those which try to separate the signal and noise subspaces, require the subspace dimension D to grow with K and N to achieve full-rank performance. Therefore, the MSWF has the advantage that for a given application, the rank can be bounded *a priori*, independent of the number of users, or data sources. The paper also evaluates the large system performance of other reduced-rank techniques by letting $(D, K, N) \to \infty$ with D/K and K/N fixed. As with prior large system analyses, the results accurately predict the performance of moderately sized systems (e.g., $N \geq 16$).

Open Problems

Our analysis of reduced-rank filters leads to some related mathematical problems, which we briefly describe. It is shown in [1] that the rank-D MSWF is the linear filter, which minimizes the output MSE subject to the constraint that the filter lies in the D-dimensional Krylov subspace spanned by the columns of $\mathbf{S}_D = [\mathbf{s}_1\ \mathbf{R}\mathbf{s}_1\ \mathbf{R}^2\mathbf{s}_1 \cdots \mathbf{R}^{D-1}\mathbf{s}_1]$, where $\mathbf{R} = E[\mathbf{r}\mathbf{r}^\dagger]$ is the input covariance matrix.[1]

Using this representation leads to an alternative expression for the large system output SINR in terms of the large system moments of $\mathbf{R}$. (See Theorem 3 in [1].) A

[1] Related filters, which are constrained to lie in a Krylov susbspace, have been proposed in [3,17–19].

direct derivation of the continued fraction formula from the large system moments of $\mathbf{R}$ has not yet been obtained. More importantly, the expression for SINR in terms of the moments of $\mathbf{R}$ applies to more general scenarios, e.g., with arbitrary powers (i.e., the mixing matrix $\mathbf{S}$ is replaced by $|\mathbf{A}|^2\mathbf{S}$ where $\mathbf{A}$ is diagonal), and multiple antennas. Generalization of the continued-fraction expression for SINR to these more general scenarios (if possible) remains an open problem. (Some progress in this direction is made in [20].)

Another problem is related to the equalization application. The model (1) again applies where $\mathbf{S}$ is Toeplitz. In that case, the "large system" output MSE corresponds to letting the filter length tend to infinity, and can be expressed in terms of the channel moments $\int_{-1/2}^{1/2} |H(e^{j2\pi f})|^{2n} df$, $n = 1, \cdots, D$, where $H(e^{j2\pi f})$ is the channel transfer function [21]. As D increases, the MSE must converge to the full-rank MMSE, given by the well-known expression $\int_{-1/2}^{1/2} \frac{\sigma^2}{\sigma^2+|H(e^{j2\pi f})|^2} df$. Analogous to the CDMA case, there is likely to be an expansion of the preceding expression, which corresponds to the performance of the reduced-rank MSWF.

A more fundamental problem is the performance of *adaptive* reduced-rank filters with finite training. That is, one of the main motivations for reduced-rank, as opposed to full-rank filtering, is that it can provide better filter estimates when training is limited. For the ideal model (2), the theory of large random matrices can again be used to evaluate large system performance, and illustrates the effect of rank selection, initialization, and data windowing on performance [22].

A related question is whether or not the performance of the adaptive MSWF with limited training is optimal in some meaningful sense. This appears to be largely unanswered. The large system analytical approach may again provide insight.

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