Correspondence

Channel Shaping to Maximize Minimum Distance

Michael L. Honig

Abstract—Suppose that N inputs to a linear, time-invariant channel are designed to maximize the minimum L_2 distance between channel outputs. It is assumed that all inputs are zero outside the finite time window $[-T, T]$ and are constrained in energy. The jointly optimal inputs and channel frequency response $H(f)$ for which the minimum distance is maximized is studied, subject to the constraint that the L_2 norm of $H(f)$ is bounded. This leads to an ellipse packing problem in which $N-1$ axis lengths, which define an ellipse in $\mathbb{R}^{N-1}$, and N points inside the ellipse are to be chosen to maximize the minimum Euclidean distance between points, subject to the constraint that the sum of the squared axis lengths is constant. An optimality condition is derived, and it is conjectured that the optimal ellipse in which the N points must lie is an n-dimensional sphere, where $n \leq N$. An approximate volume calculation suggests that n increases as $O(\log N)$. As $T \to \infty$, this implies that an optimal channel response is ideal bandlimited with bandwidth $2R'$ Hz, where $R' = (\log_e N)/(2T)$ is the information rate.

Index Terms—Signal design, minimum distance, ellipse packing.

I. Introduction

Linear equalizers in digital communications systems shape the channel frequency response to minimize distortion present in the channel output, given a particular modulation scheme. Here we investigate the possibility of simultaneously optimizing both the transmitted signals and the channel frequency response. The assumed optimality criterion is the minimum L_2 distance between channel outputs. The performance of the optimized system serves as a benchmark to which the performance of (coded) modulation schemes with suboptimal channels can be compared.

Two problems are considered. In the first, we wish to find a set of N time-limited channel inputs and a channel frequency response that jointly maximize the minimum L_2 distance between channel outputs. Here we assume that all inputs and the channel impulse response are constrained in energy (L_2). The analogous signal design problem assuming the channel response is fixed is considered in [1]. It is shown there that this (fixed-channel) problem is equivalent to packing N points in an ellipse in $\mathbb{R}^{N-1}$ so as to maximize the minimum Euclidean distance between points. The same ellipse packing interpretation applies to the problem considered here; however, in addition, the lengths of the axes of the ellipse are to be optimized in addition to the placement of the N points. In general, this packing problem is quite difficult, although our interest here is only in the shape of the ellipse, and not in the placement of points within the ellipse.

Optimality conditions are derived for the preceding packing problem, and it is conjectured that the optimal ellipse in which the points must lie is a sphere in $n < N$ dimensions. For fixed information rate $R = (\log_2 N)/(2T)$, as the length of the inputs $2T \to \infty$, the conjecture implies that an optimal channel transfer function is an ideal low-pass (or bandpass) filter. The optimized channel bandwidth depends on the asymptotic rate of growth of $n(N)$ with N. An approximate volume argument suggests that the optimized bandwidth is twice the information rate in nats/second, although currently there is no finite upper bound on this bandwidth.

In the second problem, the channel $H_0(f)$ is assumed to be fixed, and the objective is to find N time-limited channel inputs and a filter $G(f)$ that jointly maximize the minimum distance between the outputs of $H_0(f)G(f)$, subject to energy constraints on the inputs and on the impulse response of $G(f)$. For example, $G(f)$ could be a preshaping filter at the transmitter, which does not affect the noise statistics at the receiver. This problem is approximated by an ellipse packing problem in which the optimal filter parameters are easily obtained. As $T \to \infty$, this solution implies that the optimal $G(f)$ is constant for all f such that $|H_0(f)| > \beta$, where β is independent of f, and $\beta \to 0$ as the information rate $R \to \infty$.

II. L_2/L_2 Signal Design

The following problem was posed in [2]. (The discrete-time version of this problem is considered in [1].) Given a linear, time-invariant channel $H(f)$ and information rate R (bits/second), find $N = 2^{2RT}$ inputs $u_1(t), \cdots, u_N(t)$, where R and T are chosen so that N is an integer, and where $u_i(t) = 0$ for $|t| > T$, $i = 1, \cdots, N$, to maximize the squared minimum distance between outputs

$$d^2 = \min_{i \neq j} \int_{-\infty}^{\infty} [y_i(t) - y_j(t)]^2 \, dt, \tag{1}$$

subject to the constraints

$$\int_{-T}^{T} u_i^2(t) \, dt \leq 2PT, \qquad i = 1, \cdots, N \tag{C0}$$

where the channel output is $y_i(t) = \int_{-T}^{T} h(t-s) u_i(s) \, ds$, and $h(t)$ is the channel impulse response.

Because the objective is to maximize the minimum L_2 distance between channel outputs subject to an L_2 constraint on the channel inputs, we refer to this type of signal design as L_2/L_2 *signal design.* (Of course, norms other than the L_2 norm may be appropriate in some situations. A more detailed discussion is given in [1].)

It is shown in [2] and [1] that the preceding problem is equivalent to the following discrete packing problem. Find vectors $\boldsymbol{u}_i \in \mathbb{R}^{N-1}$, $i = 1, \cdots, N$, to

$$\max \left\{ d^2 = \min_{i \neq j} \|\Lambda^{1/2}(\boldsymbol{u}_i - \boldsymbol{u}_j)\|^2 \right\} \tag{P0}$$

subject to

$$\|\boldsymbol{u}_i\|^2 \leq 2PT, \qquad i = 1, \cdots, N \tag{C1}$$

where $\Lambda = \text{diag}[\lambda_1, \lambda_2, \cdots, \lambda_{N-1}]$ and $\lambda_i^{1/2}$, $i = 1, \cdots, N-1$ are the singular values of the channel. That is, $\int_{-T}^{T} R(t-s)\phi_i(s) \, ds = \lambda_i \phi_i(t)$, and $R(t) = \int_{-\infty}^{\infty} h(s)h(s+t) \, ds$, where $\phi_i(t)$ is the eigenfunction associated with λ_i. We will assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$.

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The author is with Bellcore, Morristown, NJ 07960.

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If we define $y_i = \Lambda^{1/2} u_i$, then the preceding problem is equivalent to maximizing the minimum Euclidean distance of the set $\{y_i\}$ subject to the constraint $\|\Lambda^{-1/2} y_i\|^2 \le 2PT$. That is, the vectors y_i must lie in an ellipse in $\mathbb{R}^{N-1}$ with axes having lengths $\sqrt{2PT\lambda_i}$, $i = 1, \cdots, N-1$. A particular u_i is said to lie in the "input signal space," and a particular y_i lies in the "output signal space." Similarly, a set of N vectors u_i (y_i), $i = 1, \cdots, N$ will be referred to as an "input (output) signal set." A signal set which is a solution to (P0)–(C1) will be referred to as an "optimal signal set."

A. Channel Shaping Problems

Referring to Fig. 1, suppose that at the transmitter (or receiver), we add a filter $G(f)$, which changes the shape of the effective channel frequency response from $H_0(f)$ to $G(f)H_0(f)$. Note that by adding the filter $G(f)$ at the transmitter side, the noise statistics at the receiver remain the same.

It is of interest to determine both the optimal channel shape $G(f)H_0(f)$ and the optimal $G(f)$ for fixed $H_0(f)$, subject to suitable constraints. Here we assume an energy constraint on the impulse response of the filter to be optimized. Specifically, there are two problems of interest. The first is

$$\max_{\substack{u_1, \cdots, u_N \\ H(f)}} d^2 \tag{P1}$$

subject to the input constraints (C0) and the channel constraint

$$\int_{-\infty}^{\infty} |H(f)|^2 \, df \le 1. \tag{C2}$$

In this case, $H(f) = G(f)H_0(f)$, and the problem is to jointly select the channel inputs as well as the channel transfer function to maximize the minimum distance d^2 given by (1).

The second problem is

$$\max_{\substack{u_1, \cdots, u_N \\ G(f)}} d^2 \tag{P2}$$

subject to the input constraints (C0) and the filter constraint

$$\int_{-\infty}^{\infty} |G(f)|^2 \, df \le 1 \tag{C3}$$

where the channel output $y_i(t) = g * h_0 * u_i(t)$, "$*$" denotes convolution, and $g(t)$ and $h_0(t)$ are the impulse responses of $G(f)$ and $H_0(f)$, respectively. The second problem is therefore to jointly optimize the channel inputs as well as the transmitter filter $G(f)$ for fixed $H_0(f)$.

In what follows, we first discuss only the first channel shaping problem (P1)–(C0)–(C2). Since the arguments that apply to the first problem generally also apply to the second, discussion of the second problem is postponed to a later section.

In analogy with the L_2/L_2 signal design problem (P0)–(C1), we can relate the continuous-time channel shaping problem (P1)–(C0)–(C2) to a discrete packing problem. Upon examining problem (P0)–(C1), it becomes clear that the channel affects the minimum distance only through its $N-1$ largest singular values $\lambda_1, \cdots, \lambda_{N-1}$. That is, any $H(f)$ with the same set of $N-1$ largest singular values gives the same max–min distance (P0) corresponding to an optimal signal set. Consequently, (P1)–(C0)–(C2) is equivalent to a packing problem in which d given in (P0) is to be maximized over both the vectors $u_1, \cdots, u_{N-1}$ and the channel parameters, or axis lengths $\lambda_1, \cdots, \lambda_{N-1}$. From [3, Theorem 8.4.1],

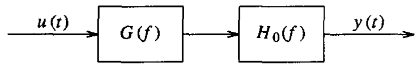

Fig. 1. Channel transfer function $H_0(f)$ with channel shaping filter $G(f)$.

$$\int_{-\infty}^{\infty} |H(f)|^2 \, df = \frac{1}{2T} \sum_{i=1}^{\infty} \lambda_i = 1. \tag{2}$$

The problem (P1)–(C0)–(C2) is therefore equivalent to the following problem:

$$\max_{\substack{u_1, \cdots, u_N \\ \Lambda}} \left\{ d^2 = \min_{i \ne j} \|\Lambda^{1/2}(u_i - u_j)\|^2 \right\} \tag{P3}$$

subject to the constraints (C1) and

$$\sum_{i=1}^{N-1} \lambda_i = 2T. \tag{C4}$$

Notice that (2) and the constraint (C4) imply that $\lambda_i = 0$ for $i > N-1$. That is, because the max–min distance is a nondecreasing function of the axis lengths, and because the N signal points span $N-1$ dimensions, the minimum distance is maximized by confining the channel impulse response to no more than $N-1$ dimensions. The optimal autocorrelation function can therefore be written as

$$R(t,s) = \sum_{i=1}^{N-1} \lambda_i \phi_i(t) \phi_i(s) \tag{3}$$

for some orthonormal set of functions $\{\phi_i(t)\}$ (see [3, Theorem 8.4.1]).

III. Optimality Conditions

The preceding packing problems can be viewed as nonlinear optimization problems for which optimality conditions can be derived. This is complicated, however, by the fact that the max–min distance criterion is not differentiable. One technique for circumventing this problem is to approximate the max–min distance d by a potential function f which is a smooth function of the parameters to be optimized. For example, one possibility is to let

$$f(u_1, \cdots, u_N; \Lambda) = -\frac{1}{K} \log_e \left(\sum_{i \ne j} e^{-K\|\Lambda^{1/2}(u_i - u_j)\|^2} \right). \tag{4}$$

Note that $\lim_{K \to \infty} f = d^2$. We now proceed to derive optimality conditions for the analogous packing problem to (P3)–(C1)–(C4), in which f replaces d as the cost criterion. The Lagrangian in this case is

$$L(u_1, \cdots, u_N; \Lambda) = f - \sum_{i=1}^{N} \mu_i \|u_i\|^2 - \xi \sum_{i=1}^{N-1} \lambda_i \tag{5}$$

where the μ_i's and ξ are Lagrange multipliers. Because d is not a differentiable function of the inputs, the derivative $\nabla_{u_i} \lim_{K \to \infty} L$ does not exist in general. However, interchanging the limit and derivative operation, that is, setting $\nabla_{u_i} L = 0$ (the $N-1$-vector of all zeros) and $\partial L / \partial \lambda_i = 0$ for $i = 1, \cdots, N-1$, and subsequently letting $K \to \infty$ gives the conditions

$$\sum_{j \in J_i} \Lambda(u_i - u_j) = \mu_i u_i, \qquad i = 1, \cdots, N \qquad \text{(OC1)}$$

$$\sum_{i=1}^{N} \sum_{j \in J_i} ([u_i]_k - [u_j]_k)^2 = \xi, \qquad k = 1, \cdots, n \qquad \text{(OC2)}$$

where the sets of indexes J_i, $i = 1, \cdots, N$ are nearest neighbor sets, i.e.,

$$J_i = \{j : \|u_j - u_i\| = d\}, \qquad i = 1, \cdots, N \qquad (6)$$

and $n \le N - 1$ is the number of dimensions spanned by the inputs u_i.

For any fixed K, the preceding procedure can be used to obtain necessary conditions for solutions to the modified packing problem in which f is maximized subject to the constraints (C1) and (C4). Because of the preceding interchange of limit with derivative, (OC1)–(OC2) are not necessary conditions for solutions to (P3)–(C1)–(C4), although (OC1)–(OC2) are optimality conditions in the following sense.

Theorem: There exists at least one solution to (P3)–(C1)–(C4) that satisfies (OC1)–(OC2).

Proof: The following argument has been previously used in [4] to prove a similar result. For fixed K and N, and for any fixed signal set $u_1, \cdots, u_N$ and singular value matrix Λ, it is easily shown that

$$f(u_1, \cdots, u_N; \Lambda) \le d^2(u_1, \cdots, u_N; \Lambda). \qquad (7)$$

Let $f_{\max}(K)$ be the maximum value of f, given the constraints (C1) and (C4), and let d_f be the corresponding minimum distance of any signal set that achieves $f_{\max}$. Also, let $d_{\max}$ be the maximum achievable d in (P3)–(C1)–(C4), and let $f_d(K)$ be the value of f in (4) evaluated for any particular signal set that achieves $d_{\max}$. Then, (7) implies that for all K,

$$f_d(K) \le f_{\max}(K) \le d_f^2(K) \le d_{\max}^2, \qquad (8)$$

and letting $K \to \infty$ implies that $f_d(K) \to d_{\max}^2$. Consequently, as $K \to \infty$, the minimum distance d_f corresponding to a signal set that achieves $f_{\max}$ converges to $d_{\max}^2$, and the corresponding signal sets satisfy (OC1)–(OC2). □

The condition (OC1) was previously derived in [1], and applies to (P0)–(C1) with fixed λ_i's. The condition (OC2) governs the optimal choice of λ_i's (i.e., the channel characteristics), and states a symmetry condition for the signal constellation about each axis.

A. Example: $N = 3$

To illustrate the preceding discussion, consider problem (P3)–(C1)–(C4) when $N = 3$. That is, we wish to place three points within an ellipse in $\mathbb{R}^2$ to maximize d. A solution for a particular ellipse in the output space is shown in Fig. 2. In this case, y_1 lies along the y-axis, and y_2 and y_3 lie where the ellipse intersects a circle centered at y_1 with radius d. Transforming the space by the operator $T: y \to \Lambda^{-1/2} y$ gives the corresponding input constellation shown to the right in Fig. 2. It is easily verified that if the eccentricity of the ellipse is sufficiently large, then this is indeed the solution to (P0)–(C1), and the constellation points satisfy the optimality condition (OC1).

For any ellipse in $\mathbb{R}^2$ with the major and minor axes aligned along the x- and y-axes, respectively, it is apparent that a solution to the packing problem (P0)–(C1) must have the following properties: 1) one point (y_1) lies at the top (or bottom) of the minor axis, and 2) the remaining two points (y_2 and y_3) are symmetrically placed about the y-axis. Taking $2PT = 1$ in (C1), this implies that the input constellation satisfies

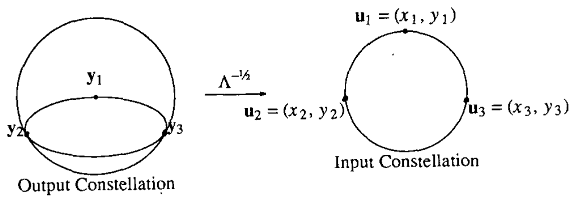

Fig. 2. Particular solution to (P0)–(C1) for $N = 3$.

$$(x_1, y_1) = (0, 1), \qquad x_2 = -x_3, \qquad y_2 = y_3 \qquad (9)$$

where $u_i = (x_i, y_i)$.

We now use (OC2) to obtain an additional condition on the optimal shape of the ellipse. It is necessary, however, to assume a set of nearest neighbor points J_i for each point u_i. There are only two possible choices in this case. The first is that $J_1 = \{u_2, u_3\}$ and $J_2 = J_3 = \{u_1\}$. The second choice is that the points u_i are equidistant from each other, that is, $J_i = \{u_j, j \ne i\}$ for each $i = 1, 2, 3$. For the first choice of nearest neighbors, (OC2) implies that

$$(x_1 - x_2)^2 + (x_1 - x_3)^2 = (y_1 - y_2)^2 + (y_1 - y_3)^2. \quad (10)$$

Combining (9), (10), and the constraint $x_i^2 + y_i^2 = 1$, $i = 1, 2, 3$ implies that $y_2 = y_3 = 0$. Consequently, there exists a solution to the optimality conditions (OC1)–(OC2) for which $u_1 = (0, 1)$, $u_2 = (-1, 0)$, and $u_3 = (1, 0)$, as shown in Fig. 3. Clearly, this packing maximizes d only if $\lambda_2 = 0$ and $\lambda_1 = 2T$. That is, the output points are collinear with $d = \sqrt{2T}$, as shown to the right in Fig. 3.

For the second choice of nearest neighbor sets, the input constellation must consist of the vertices of an equilateral triangle inscribed by the circle representing the input constraints. It is easily verified that this packing is optimal and satisfies (OC1)–(OC2) if $\lambda_1 = \lambda_2 = T$. In this case, $d = \sqrt{6T}$, which is larger than the distance which resulted from the first assumption on nearest neighbor sets. Consequently, we conclude that a solution to (P3)–(C1)–(C4) for $N = 3$ consists of a simplex in $\mathbb{R}^2$, and that the optimal ellipse (in the output space) is a circle. Because the number of possible nearest neighbor sets grows extremely fast with the number of points in the constellation, obtaining solutions to (OC1)–(OC2) analytically becomes quite difficult for $N > 3$.

IV. Conjectured Solution

The following conjecture states that the optimal boundary ellipse (in the output space) is an n-dimensional sphere of radius $r = 2T\sqrt{P/n}$ for some $n < N$.

Conjecture: There exists at least one solution to (P3)–(C1)–(C4) for which

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n, \qquad n < N \qquad \text{(11a)}$$

and

$$\lambda_m = 0, \qquad m > n. \qquad \text{(11b)}$$

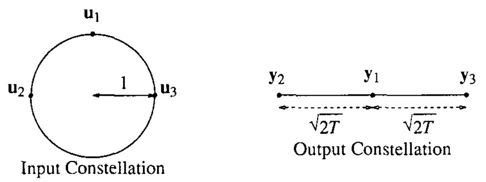

Fig. 3. Solution to (OC1)–(OC2) for $N = 3$ and $2PT = 1$.

It can be shown that the conjecture is consistent with (OC1)–(OC2) for at least one n. That is, (OC1)–(OC2) are satisfied by an N-point simplex in $n = N - 1$ dimensions, and by placing points at the end of each axis of an $n = N/2$-sphere. In the latter case, $d = 4T\sqrt{P/N}$. One method by which the conjecture might be proved is to show that (OC1)–(OC2) do not have a solution if $\lambda_i \neq \lambda_j$, where $\lambda_i > 0$ and $\lambda_j > 0$. Whether or not this is true remains an open question.

The conjecture is supported by the following elementary volume argument. Rather than maximize minimum distance d for fixed N, as stated in the previous problems, we can alternatively maximize the number of points for which the minimum distance is at least d. That is, let

$$N_{\max}(T, d) = \max\left\{N: \min_{i \neq j} \|y_i(\cdot) - y_j(\cdot)\| \geq d\right\}. \quad (12)$$

For a channel with singular value matrix Λ, we can approximate $N_{\max}(T, d)$ as the volume of the ellipse in which the points lie divided by the volume of a sphere of radius $d/2$. The dimension of the signal set is $n < N_{\max}$, and should be chosen to maximize the resulting estimate. That is, letting γ_n be the volume of the n-sphere (sphere in $\mathbb{R}^n$) with radius one, we have that

$$N_{max}(T, d) \approx \max_{n \leq N-1}\left\{\frac{\gamma_n \prod_{i=1}^{n}(2PT\lambda_i)^{1/2}}{\gamma_n (d/2)^n}\right\}$$
$$= \max_n \left\{\left(\frac{\sqrt{8PT}}{d}\right)^n \prod_{i=1}^{n}\sqrt{\lambda_i}\right\}. \quad (13)$$

Choosing the λ_i's to maximize this expression subject to the constraint (C4) gives

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{2T}{n}, \quad (14)$$

and substituting into the approximation for $N_{\max}$ implies that the maximum d for given N is given by

$$d^2_{\max} \approx \max_n \frac{16PT^2}{nN^{2/n}} = \frac{4PT}{eR'} \quad (15)$$

where

$$R' = \frac{\log_e N}{2T} = \frac{R}{\log_2 e} \quad (16)$$

since the maximum in (15) occurs at $n = 2\log_e N$.

Assuming the conjecture is true, it follows that

$$\max_n \left(\frac{r}{d}\right)^n \leq N_{\max}(T, d) \leq \max_n \left(1 + \frac{2r}{d}\right)^n \quad (17)$$

where $r = 2T\sqrt{P/n}$ is determined by the constraints (C1) and (C4). The lower bound is simply the volume of an $\underline{n}$-sphere of radius r divided by the volume of an $\underline{n}$-sphere of radius d, where $\underline{n}$ is the dimension for which the max occurs. This lower bound is valid whether or not the conjecture is true. The upper bound is the volume of an $\bar{n}$-sphere of radius $r + d/2$ divided by the volume of an $\bar{n}$-sphere of radius $d/2$, where $\bar{n}$ is the n for which the max occurs.

If the restriction that $\underline{n}$ be an integer is ignored, then $\underline{n} = 4T^2P/(d^2e)$. In contrast, the upper bound in (17) is an increasing function of n. Notice, however, that n must be chosen so that $d/2 \leq r$ is satisfied. With this restriction, it follows that $\bar{n} = 16T^2P/d^2$. Substituting these values of n in (17) gives

$$\frac{PT}{eR'} \leq d^2_{\max} \leq \frac{8PT}{R}. \quad (18)$$

The conjecture therefore implies that the max–min squared distance increases linearly with T.

For any number of messages $N \geq 2$, we can define $n^*(N)$ as the dimension n in (11) for which the max–min distance between the N points is maximized. That is, according to the conjecture, we wish to pack the N points in an n-sphere of radius $r = 2T\sqrt{P/n}$, which we denote as $S_n(r)$. Letting

$$\tilde{d}_n(N) = \max_{\substack{u_i \in S_n(r) \\ i = 1, \cdots, N}} d(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N; n), \quad (19)$$

then by definition,

$$\max_n \tilde{d}_n(N) = \tilde{d}_{n^*}(N). \quad (20)$$

As an example, $n^*(2) = 1$, and it is easily verified that $n^*(3) = n^*(4) = 2$. The preceding volume estimate (15) suggests that $n^*(N)$ increases as $O(\log N)$. Unfortunately, this type of argument is too crude to provide useful upper or lower bounds on the asymptotic growth rate of $n^*(N)$.

Assuming the conjecture is true, as $T \to \infty$, an optimal channel transfer function can be determined by applying Szegö's Theorem (see [3, Lemma 8.5.2]). Specifically, for any real, even, absolutely integrable function $R(t)$ with absolutely integrable Fourier transform $F(f)$, let $N_{2T}(a, b)$ denote the number of squared singular values of $F(f)$ satisfying $a \leq \lambda_i^F < b$, where $b \geq a > 0$. If $\text{meas}(\{f: F(f) = a\} \cup \{f: F(f) = b\}) = 0$, then Szegö's Theorem states that

$$\lim_{T \to \infty} \frac{1}{2T} N_T(a, b) = \int_{a \leq F(f) < b} df. \quad (21)$$

For any fixed T and small $\epsilon > 0$, the conjecture states that

$$\frac{1}{2T} N_{2T}(0, 1 - \epsilon) = \frac{1}{2T} N_{2T}(1 + \epsilon, \infty) = 0 \quad (22)$$

and

$$\frac{1}{2T} N_{2T}(1 - \epsilon, 1 + \epsilon) = \frac{n^*(N)}{2T} \quad (23)$$

where $N = 2^{2RT}$. A channel transfer function that has this asymptotic distribution of eigenvalues is

$$H(f) = \begin{cases} \dfrac{1}{2W} & f \in B \\ 0 & f \notin B \end{cases} \tag{24}$$

where meas $B = 2W$ and

$$2W = \lim_{T \to \infty} \frac{n^*(N)}{2T}. \tag{25}$$

The volume approximation for $d_{\max}$ given by (15) suggests that $n^*(N) \approx 2 \log_e N$ as $T \to \infty$, which would imply that the bandwidth $2W \approx 2R'$. However, as stated previously, useful upper and lower bounds on the asymptotic growth rate of $n^*(N)$, which would yield useful bounds on W, have not been obtained.

V. Optimization of Transmitter Filter for Fixed $H_0(F)$

We now consider the second problem stated in Section II-A in which the transmitter (or receiver) filter is optimized for fixed $H_0(f)$. From the discussion in Section II, it is apparent that the continuous-time problem (P2)–(C0)–(C3) is equivalent to the following finite-dimensional packing problem:

$$\max_{\substack{u_1, \cdots, u_N \\ \Lambda^G}} \left\{ d^2 = \min_{i \neq j} \|(\Lambda^{GH})^{1/2}(\boldsymbol{u}_i - \boldsymbol{u}_j)\|^2 \right\} \tag{P4}$$

subject to the constraints (C1) and

$$\sum_{i=1}^{N-1} \lambda_i^G = 2T \tag{C5}$$

where the superscripts "GH" and "G" signify quantities associated with the channels $G(f)H_0(f)$ and $G(f)$, respectively.

Because Λ^{GH} is a very complicated function of Λ^G for any fixed T, problem (P4)–(C1)–(C5) is not a very useful restatement of the continuous-time problem (P2)–(C0)–(C3). However, for large T, the singular-value matrix Λ^{GH} can be approximated by $\Lambda^G \Lambda^H$. That is, from Szegö's Theorem (21), it is straightforward to show that

$$\lim_{T \to \infty} \frac{1}{2T} \sum_{i=1}^{n_H^*(N)} (\lambda_i^{GH} - \lambda_i^G \lambda_i^H)^2 = 0 \tag{26}$$

provided that $|G(f)|$ and $|H(f)|$ are piecewise continuous and bounded, where $n_H^*(N)$ is the number of dimensions spanned by an optimal signal set for the channel $H(f)$ containing $N = 2^{2RT}$ signals.

If we define the approximation error for a given optimal signal set $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_N$ as

$$\begin{aligned} \epsilon &= \min_{i \neq m} \|(\Lambda^{GH})^{1/2}(\boldsymbol{u}_i - \boldsymbol{u}_m)\|^2 - \min_{i \neq m} \|(\Lambda^G \Lambda^H)^{1/2}(\boldsymbol{u}_i - \boldsymbol{u}_m)\|^2 \\ &= \min_{i \neq m} \sum_k (\lambda_k^{GH} - \lambda_k^G \lambda_k^H)(u_{ik} - u_{mk})^2 \end{aligned} \tag{27}$$

where $u_{ik} = [\boldsymbol{u}_i]_k$, then it seems difficult to show that $|\epsilon|/d_{\max}^2 \to 0$ as $T \to \infty$. However, if we approximate Λ^{GH} in (P4) by $\Lambda^G \Lambda^H$, then optimizing the filter $G(f)$ in this case becomes equivalent to maximizing $\prod_{i=1}^{n_H^*(N)} \lambda_i^G$ subject to the constraint (C5). In this case, $\lambda_i^G = 2T/n_H^*(N)$ for $i = 1, \cdots, n_H^*(N)$, and $\lambda_i = 0$ for $i > n_H^*(N)$. As $T \to \infty$, Szegö's Theorem then implies that the measure of the set of frequencies for which $G(f) \neq 0$ is given by

$$\mu = \lim_{T \to \infty} \frac{n_H^*(N)}{2T}. \tag{28}$$

Furthermore, since by assumption the largest singular values of $G(f)$ multiply the largest singular values of $H(f)$, i.e., $\lambda_i^{GH} \approx \lambda_i^G \lambda_i^H$, $i = 1, \cdots, n_H^*(N)$, Szegö's Theorem implies that $G(f) = 1/\mu$ for $f \in B(f)$, and $G(f) = 0$ for $f \notin B(f)$, where

$$B(f) = \{f: |H_0(f)| > \beta\}, \tag{29}$$

and β is selected so that meas $B(f) = \mu$.

The asymptotic spectrum of optimal signal sets for a given channel $H_0(f)$ as $T \to \infty$ is defined in [1], and the discussion there indicates that $B(f)$ is the set of frequencies for which the asymptotic spectrum is nonzero. Furthermore, β decreases with R, and $R \to \infty$ implies that $\beta \to 0$. We therefore conclude, subject to the accuracy of the preceding approximation, that when the inputs are selected to maximize minimum distance, the optimum filter $G(f)$ as $T \to \infty$ simply scales the transmitted spectrum. No additional improvement in minimum distance can therefore be obtained by choosing $G(f)$ to change the shape of the channel frequency response.

VI. Conclusions

The problem of jointly selecting optimal transmitted signals and the channel frequency response, assuming the channel is linear and time-invariant, has been considered where the minimum L_2 distance between channel outputs is the optimization criterion. Results indicate that for a fixed information rate, as the length of the inputs goes to infinity, the optimal channel frequency response is a constant wherever it is positive; however, a proof of this does not currently exist. A crude but simple volume estimate suggests that the optimal channel bandwidth ($2W$) is equal to twice the information rate ($2R'$). However, there currently is no finite upper bound on this optimal bandwidth.

A second problem considered is the joint optimization of channel inputs and a transmitter filter given a fixed channel response. This problem also remains unsolved in general, although the discussion in the preceding section indicates that the optimal filter transfer function should simply be constant over the band where the asymptotic transmitted spectrum is nonzero. This seems intuitively reasonable since if the transmitted signals are optimized, then presumably, the transmitted spectrum is also optimized, eliminating the potential benefits of adding a filter at the transmitter or receiver.

There are, of course, additional variations on the problems presented here that can be considered. As an example, one might choose to constrain the inputs to the channel in amplitude rather than energy, and/or constrain the channel impulse response in a different way. Different criteria (i.e., norms) for separating the channel outputs are also possible. However, in cases of interest, it seems likely that the associated packing problem will turn out to be intractable.

References

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