Sequential Bandwidth and Power Auctions for Spectrum Sharing

Junjik Bae, Eyal Beigman, Randall Berry, Michael L. Honig, and Rakesh Vohra

Abstract

We study a sequential auction for sharing a wireless resource (bandwidth or power) among competing transmitters. The resource is assumed to be managed by a spectrum broker (auctioneer), who collects bids and allocates discrete units of the resource via a sequential second-price auction. It is well known that a second price auction for a single indivisible good has an efficient dominant strategy equilibrium; this is no longer the case when multiple units of a homogeneous good are sold in repeated iterations. For two users with full information, we show that such an auction has a unique equilibrium allocation. The worst-case efficiency of this allocation is characterized under the following cases: (i) both bidders have a concave valuation for the spectrum resource, and (ii) one bidder has a concave valuation and the other bidder has a convex valuation (e.g., for the other user's power). Although the worst-case efficiency loss can be significant, numerical results are presented, which show that for randomly placed transmitter-receiver pairs with rate utility functions, the sequential second-price auction typically achieves the efficient allocation. For more than two users it is shown that this mechanism always has a pure strategy equilibrium, but in general there may be multiple equilibria. We give a constructive procedure for finding one equilibrium; numerical results show that when all users have concave valuations the efficiency loss decreases with an increase in the number of users.

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I. INTRODUCTION

In some dynamic spectrum sharing scenarios a spectrum owner, or licensee may wish to lease spectrum to secondary users (e.g., see [1]–[3], which discuss secondary spectrum markets).

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Junjik Bae, Randall Berry and Michael Honig are with Department of Electrical Engineering and Computer Science at Northwestern University (e-mails: junjik@u.northwestern.edu, {rberry, mh}@eecs.northwestern.edu).

Eyal Beigman and Rakesh Vohra are with CMS-EMS, Kellogg School of Management at Northwestern University (e-mails: {e-beigman, r-vohra}@northwestern.edu).

There are various scenarios for how this can be accomplished. We focus on the scenario where a *spectrum manager*, or *broker* is responsible for allocating spectrum usage among non-cooperative secondary users. Examples of this scenario are also presented in [4]–[6]. Other distributed spectrum sharing scenarios, which do not rely on the presence of a spectrum manager, are considered in [7]–[9].

The spectrum manager can mitigate the effects of externalities (interference) and increase the overall efficiency by soliciting information about user utilities and channel conditions. This is naturally accomplished via an auction. We consider such an approach where n discrete units of either bandwidth or power are auctioned to users in a peer-to-peer wireless network. In the bandwidth auction each unit of bandwidth is allocated to a particular user, so that the users do not interfere. In the power auction all users spread their power across the same band and each user's power is viewed as an *individual* resource. The power auction then applies to the situation in which a user wishes to acquire power by bidding against other users, who wish to reduce their interference.

Numerous auction mechanisms can be applied to this scenario. Of these, it is well-known that the Vickrey-Clarke-Groves (VCG) mechanism achieves the efficient outcome.[1] However, there are pragmatic reasons to prefer alternative mechanisms. We consider one such mechanism here, namely a *sequential second price auction*. In this mechanism, each resource unit is auctioned off sequentially according to a second-price auction.[2] Sequential auctions have been used in many applications (e.g., see [10]–[14]) since they require relatively little computation and information exchange among the agents and the broker, compared with other mechanisms. In addition, sequential auctions easily accommodate scenarios in which agents enter and leave the market at arbitrary times, and allow the broker to allocate resources incrementally. However, it is well known that sequential auctions do not always achieve an efficient allocation [13]. In this paper, we study this efficiency loss.

There is an extensive literature that investigates the properties of sequential auctions [13]–[21] assuming that valuations are private information. Since the assumption of private information

[1]For a given auction mechanism, the bidders can be viewed as playing a game, in which their actions are their bids. The auction is efficient if the equilibrium of this game maximizes the total utility of the agents.

[2]Namely, each unit is allocated to the highest bidder, who pays the second-highest bid.

complicates the analysis, those papers restrict attention to the case of bidders with *unit* demands and in some cases to just two bidders. Here we allow bidders to have multi-unit demands, but for tractability assume full information. Abstracting away from private information allows us to focus on the strategic implications of bidding in sequential auctions. Indeed, in work such as [14], the efficiency loss is due to the information asymmetries and not the mechanism. If agents in that model have full information (each with unit demand), then it can be shown that the auction achieves an efficient outcome. We also note that assuming full information is consistent with prior work, such as [22], [23], which also study the efficiency loss of different mechanisms.

For two users and an arbitrary number of resource units, our results show that the sequential second price auction always has a unique equilibrium[3] allocation. We characterize the worst-case efficiency loss of this equilibrium for the case where each agent has a concave utility for the spectrum resource, and the case where one agent has a concave utility and the other agent has a convex utility. The former case models the bandwidth auction, while the latter can arise in the power auction, when a user bids to reduce interference.

We also present simulation results for the efficiency loss when the two users' channel gains are randomly generated. The utility function for each user is the maximum achievable (Shannon) rate, where interference is treated as background noise. The results show that except for a small fraction of realizations, the equilibrium allocation is efficient. Furthermore, we show that the worst-case efficiency for the bandwidth auction improves due to constraints on the marginal utilities imposed by this utility function.

For more than two users, each with a concave utility function, we show that the auction has at least one pure strategy equilibrium. Furthermore, the equilibrium allocation may not be unique. Hence, some coordination of the users may be required to decide on a particular outcome. This makes characterizing the efficiency loss more difficult. Numerical results show that the empirical distribution of the efficiency for the bandwidth auction with three users is stochastically better than that with two users. This suggests that the worst-case efficiency loss is attained with two users.

[3] See Section III for a precise definition of the equilibrium concept we use.

II. Spectrum Sharing Model

We consider a model for spectrum sharing among k users, where each user consists of a distinct transmitter-receiver pair. As in [7], [8], we model this as a k-user Gaussian interference channel with frequency flat fading. The channel gain between user i's transmitter and user j's receiver is denoted by h_{ij}. Each transmitter has an average power constraint P, and the total available bandwidth is W Hz. We further assume that each transmitter uses an optimal (capacity achieving) code, where the received interference is treated as background noise (i.e., no interference cancellation is used).

We focus on two spectrum sharing techniques: frequency division multiplexing (FDM) and spread-spectrum signaling with frequency-flat power allocations across the entire band.[4] With FDM, each user i receives bandwidth W_i, where $\sum_j W_j = W$, resulting in the achievable rate

$$r_i(W_i) = W_i \log(1 + \tfrac{h_{ii}P}{N_0 W_i}), \tag{1}$$

where N_0 is the power spectral density of the additive noise. With full spreading, user i receives power $P_i \in [0, P]$ resulting in the achievable rate

$$r_i(P_i, P_{-i}) = W \log(1 + \tfrac{h_{ii}P_i}{N_0 W + \sum_{j \neq i} h_{ji} P_j}), \tag{2}$$

where P_{-i} is the vector of powers of each user except i. Each agent is endowed with a utility function, $U_i(r_i)$, which is increasing and concave.

Assume that agent 1 is initially using the spectrum with a given bandwidth or power allocation, and agents $2, \ldots, k$ want to share this spectrum. The spectrum manager divides the appropriate resource into n units and re-allocates these units among the k agents. In the FDM case, each resource unit represents a frequency band of W/n Hz. The manager either re-allocates a unit to some agent i or lets agent 1 continue to use that unit. Let u_s^i be agent i's marginal valuation for receiving her s-th unit, i.e., $u_s^i = U_i(r_i(sW/n)) - U_i(r_i((s-1)W/n))$. From (1), it follows that the marginal valuations of each agent are decreasing, i.e. $u_1^i \geq \ldots \geq u_n^i$.

In the full-spread case, the manager allows each agent $i \neq 1$ to continue transmitting at its current power, P_i, and only allocates the power of agent 1 ($P_1 = P$) among all agents. Each

[4] More generally, the users could each pick a power allocation over frequency; our choices represent two specific classes of power allocations. Restricting ourselves to these classes simplifies resource allocation. Furthermore, for many choices of channel gains, the optimal power allocation is in this set [7].

alloted unit represents a power increment of P/n. A unit allocated to agent 1 allows her to increase her transmission power, whereas a unit allocated to agent $i \neq 1$ decreases the power assigned to agent 1, thereby reducing agent i's interference. Agent 1's marginal valuation for the s-th unit is $u_s^1 = U_1(r_1(sP/n, P_{-1})) - U_1(r_1((s-1)P/n, P_{-1}))$, which is again decreasing in s. On the other hand, the valuation for agent $i \neq 1$ depends on how many units she receives as well as how many units agent 1 receives (which increases her interference).[5] For two agents, given that agent 2 receives s units at the end of the auction, agent 1 must receive $n-s$ units. Therefore, agent 2's marginal valuations are given by $u_s^2 = U_2(r_2(P_2, (n-s)P/n)) - U_2(r_2(P_2, (n-s+1)P/n))$, which is not necessarily decreasing in s. For example, if $U_2(r_2)$ is linear, then u_s^2 is increasing in s.[6] Here, we focus on the case of two agents, which corresponds to the case where a node has only one dominant interferer. For more than two agents, agent $i \neq 1$'s valuation can not be written in terms of only her allocation, which complicates the analysis.[7]

III. Sequential Second-Price Auction

For a given spectrum sharing technique, we consider the case where each of the n resource units are allocated among k agents via a sequential second-price auction. In this auction, the units are allocated sequentially in n rounds. In round $m \leq n$, each agent submits a bid for the mth unit. The auctioneer allocates this unit to the agent with the largest bid and charges that agent the second largest bid. We refer to this as a bandwidth (power) auction in the FDM (full-spread) case.

This mechanism can be viewed as an extensive form game with a balanced k-ary game tree. Each decision node in the game tree designates a state of the world, where a certain quantity of goods (resource units) are allocated to agents $1, \ldots, k$. Let $\mathbf{s} = (s_1, \ldots, s_k)$ denote such an allocation. Since the goods are homogeneous, the decision nodes with the same allocation

[5] Note in the bandwidth allocation, the marginal values of one agent does not depend on how many units any other agent receives.

[6] Indeed user 2's marginal valuations may be neither increasing nor decreasing for all s. In general this depends on the utility, the choice of channel gains and the power levels. A necessary condition for the marginals to be increasing or decreasing can be given in terms of the utility's coefficient of relative risk aversion, as in [8].

[7] In particular for more than two agents whenever an agent $i \neq 1$ is allocated a unit, it decreases the interference for all agents $j \neq 1$ thus providing agents an incentive to "free-ride" on each other.

can be unified and the game tree can be replaced with a directed graph $G = (V, E)$, where $V = \{\mathbf{s} \in [1, \ldots, n]^k | \sum_{i=1}^{k} s_i \leq n\}$ (see Fig. 1). A node $\mathbf{s} \in V$ represents the outcome of the $(\sum_{i=1}^{k} s_i)$-th round, in which agent i has been allocated s_i. For $\sum_{i=1}^{k} s_i < n$, each node $\mathbf{s}$ has directed edges to k children $(s_1, \ldots, s_i + 1, \ldots, s_k)$, $i = 1, \ldots, k$; the ith edge corresponds to agent i winning the current round. The auction begins at the root node $(0, \ldots, 0)$.

Let u_j^i denote the marginal valuation of agent i for the jth unit. Agent i's total valuation for receiving s_i units is therefore $\sum_{j=1}^{s_i} u_j^i$. Let H designate the set of observable bidding histories. A *strategy* $\sigma_i : V \times H \rightarrow \mathbb{R}^+$ is a function mapping states of the allocation and observable histories to bids. The strategy set of an agent is the set of all such functions. The outcome path of a strategy profile $\{\sigma_1, \ldots, \sigma_k\}$ is a directed path $\delta = \{\mathbf{s}^1, \ldots, \mathbf{s}^n\}$ in G such that if $s_i^{t+1} = s_i^t + 1$ and $s_j^{t+1} = s_j^t$ for $j \neq i$ then $\sigma_i(\mathbf{s}^t, \Gamma_t) \geq \sigma_j(\mathbf{s}^t, \Gamma_t)$, for all $j \neq i$, where Γ_t is the bidding history of the first t units.[8] The total payment of agent i along the path δ is $P_i(\delta) = \sum_{t=1}^{n} p_i(\mathbf{s}^t)$, where for each $\mathbf{s}^t \in \delta$, $p_i(\mathbf{s}^t) = \max\{\sigma_j(\mathbf{s}^t, \Gamma_t) : j \neq i\}$ if $s_i^{t+1} = s_i^t + 1$ and $p_i(\mathbf{s}^t) = 0$, otherwise.

We consider two types of bidding strategies: *myopic* and *sophisticated*. A *myopic* bidding strategy maximizes the immediate payoff during each round. Hence, myopic agents bid their marginal values in each round, i.e., $\sigma_i(\mathbf{s}^t, \Gamma_t) = u^k_{s_i^t + 1}$. When all agents have decreasing marginal values, myopic bidding results in an efficient outcome.[9] This strategy may be an equilibrium depending on the information structure of the extensive form game (i.e., the strategy may be rationalized [24]). For example, if the agents do not know the number of units on the market or the valuations of the other agents, it may be rational to myopically bid every round under the belief that the current round is the terminal round.[10] We will see, however, that myopic bidding is generally not a dominant strategy with full information.

A *sophisticated* bidding strategy maximizes an agent's payoff over final or expected final outcomes. The ability to make inferences on the final outcome requires that the agent be sufficiently informed about the preferences and strategies of the other agent. Here, we assume full information, i.e., each agent knows the number of units being sold, bidding histories, and the valuations of the other agent. A similar analysis could be made for the case where the agent

[8] In the case of ties, any tie-breaking rule that allocates the good to one of the agents can be used.

[9] This is not always the case if at least one agent has increasing marginal valuations.

[10] If there are no restrictions on the agent's beliefs then virtually any bidding strategy can be rationalized [24].

is Bayesian and knows the distribution of the other agent's marginal values.

IV. Analysis for Two Agents

We now consider the sequential second price auction with $k = 2$ agents and an arbitrary number of resource units. First we characterize the outcome of this auction with sophisticated bidding and full information. Since all agents know when the last unit is being sold, regardless of the bidding history, the last round of the auction is a standard second-price auction for the nth good. (The values for this good will, of course, depend on the outcomes of the previous rounds). Hence it is a weakly dominant strategy for the agents to bid their marginal values on the last round.[11] Since those values are common knowledge, all agents know beforehand the allocation and payments in the last round. Thus, we can think of the penultimate round as an auction over the right to participate in one of two auctions in the last round. Since the payoffs of each one of those auctions is common knowledge, we can think of the penultimate round as a second-price auction with valuation equal to payoff difference between those two auctions. It is therefore a weakly dominant strategy in the penultimate round to bid the payoff difference associated with the outcomes of the two auctions in the last round.

We can proceed in this way inductively until we reach the root. This shows that sophisticated bidding is the only strategy that survives iterative elimination of weakly dominated strategies.[12] This does not rule out other equilibria and in fact there may exist other Nash equilibria with higher payoffs for both agents (if, for example, they conspire against the seller). However, those equilibria must rely on unreliable threats and commitments. We eliminate those equilibria from consideration by focusing on *subgame perfect equilibria* that survive the iterative elimination of weakly dominated strategies.[13] This discussion is summarized in the following theorem.

Theorem 1: With two fully informed agents, the sophisticated bidding equilibrium is the only subgame perfect equilibria that survives iterative elimination of weakly dominated strategies.

[11] A strategy is *weakly dominant* for an agent if no other strategy gives that agent a larger pay-off, for any choice of strategies for the other agents.

[12] In other words all strategies which are weakly dominated are removed from consideration [24].

[13] A subgame perfect equilibrium is a refinement of the concept of Nash equilibrium with the restriction that agents cannot make non-credible threats [24].

We define the *equilibrium path* to be the outcome path when both agents use a sophisticated bidding strategy, and the *sequential allocation* to be the allocation at the terminal node of the equilibrium path. From the previous discussion if all agents apply a sophisticated bidding strategy, then all equilibria have the same equilibrium path, and the same (unique) sequential allocation.

Example: Consider a sequential auction with $n = 2$ units. Figure 1 (c) shows the directed graph G with each node labeled by the allocation (s_1, s_2). Assume that $u_1^1 = u_2^1 = 5$, $u_1^2 = 4$ and $u_2^2 = 1$. Since agent 1 values each unit more than agent 2 values any unit, the efficient allocation is to give both units to agent 1.

Now let us examine sophisticated bidding for this example. Assume that the game reaches node $v = (1, 0)$, so that the agents bid for the one remaining unit, given that the first unit has gone to agent 1. (See Fig. 1 (a).) In this stage it is weakly dominate for the agents to bid their valuations, i.e., agent 1 bids $u_2^1 = 5$ and agent 2 bids $u_1^2 = 4$. The auctioneer then allocates the unit to agent 1 and charges her a price of 4. Hence the value of node $v = (1, 0)$ to agent 1 is $u_1^1 + (u_2^1 - u_1^2) = 6$, where u_1^1 is the value from winning the first unit and $u_2^1 - u_1^2$ is the surplus for winning the second unit. The value of $v = (1, 0)$ to agent 2 is 0. Similarly, the value of $v = (0, 1)$ is 4 to either agent. Given these values, the agents can optimize their bids for the first unit. In particular, agent 1 bids her marginal valuation, which is $6 - 4 = 2$, and agent 2 bids $4 - 0 = 4$. It follows that agent 2 wins the first unit and pays 2. Therefore the equilibrium path is $\delta = \{(0, 0), (0, 1), (1, 1)\}$, i.e., each user receives one unit. Note that δ does not terminate in an efficient allocation. In what follows, we characterize the efficiency loss of this equilibrium.

A. *Efficiency Bound With Decreasing Marginal Values*

Given n resource units and two agents, let $(l, n-l)$ denote the efficient allocation, and $(l', n-l')$ denote the sequential allocation. The *worst-case efficiency* is defined by

$$\eta(n) = \min_{\{u_i^1\},\{u_i^2\}} \frac{\sum_{i=1}^{l'} u_i^1 + \sum_{i=1}^{n-l'} u_i^2}{\sum_{i=1}^{l} u_i^1 + \sum_{i=1}^{n-l} u_i^2}.$$

That is, the worst-case is with respect to the marginal values. We refer to $1 - \eta(n)$ as the *worst-case efficiency loss*. The next theorem characterizes $\eta(n)$ when each agent has decreasing marginal values, as in the bandwidth auction from Section II.

Theorem 2: In a two-agent sequential second-price auction with decreasing marginal values $\eta(n) \geq 1 - e^{-1}$.

In other words, the worst case efficiency loss is bounded by e^{-1}. Moreover, it can be shown that $\eta(n)$ decreases with n, and the bound $1-e^{-1}$ is asymptotically tight as $n \to \infty$.

1) Worst-Case Utility Profiles: To prove Theorem 2 we first show that the worst-case utilities have the following form.

Definition 1: Agent 1's utilities are *dominant* if $u_1^1 \geq \ldots \geq u_n^1 \geq u_1^2 \geq \ldots \geq u_n^2$. We will also refer to this as a *dominant utility profile*. Agent 1's utilities are *flat dominant* if $u_1^1 = \ldots = u_n^1 \geq u_1^2 \geq \ldots \geq u_n^2$.

The efficient allocation for a dominant utility profile is to assign all units to agent 1. In the sequential allocation, however, agent 2 may receive up to $n-1$ units.

Lemma 3: Let (s,t) be the sequential allocation. With a dominant utility profile, $s \geq 1$ and agent 1 pays u_{n-s+1}^2 for each unit she receives.

Proof: We prove this by induction on n. It is immediate for $n=1$ since the auction is then a standard second-price auction. For $n>1$ the root can be viewed as making a decision between two alternatives, namely, either agent 1 or 2 receives the first unit, and both agents then participate in an auction for $n-1$ units.

If the equilibrium path allocates the first unit to agent 2, then agent 1 pays nothing and the lemma follows by induction. This is because the equilibrium path for the n-unit auction contains the equilibrium path for the $(n-1)$-unit auction (subgame) rooted at node $(0,1)$, and the utilities associated with the subgame have a dominant profile (i.e., $u_1^1 \geq \ldots \geq u_{n-1}^1 \geq u_2^2 \geq \ldots \geq u_n^2$).

If the equilibrium path allocates the first unit to agent 1, then it suffices to show that she pays u_{n-s+1}^2. User 2 bids the difference in value between the two subgames rooted at nodes $(0,1)$ and $(1,0)$. This is the difference between participating in an $(n-1)$-unit sequential auction and receiving an extra unit, and participating in the same auction without the extra unit. This difference is therefore the value of the extra unit, u_{n-s+1}^2, which is agent 1's payment. ■

Under the assumptions of Lemma 3, we can write the pay-off of agent 1 for any terminal allocation (s,t), assuming that this is the sequential allocation. Furthermore, since agent 1's utility is dominant, she can choose the terminal allocation, which gives her the highest pay-off. Her choice will be the sequential allocation. This is summarized in the following corollary.

Corollary 4: Given a dominant utility profile, the allocation (s,t) is the sequential allocation if and only if

$$\sum_{i=1}^{s}(u_i^1 - u_{n-s+1}^2) \geq \sum_{i=1}^{r}(u_i^1 - u_{n-r+1}^2),\ \forall r \in \{1, \dots, n\}. \tag{3}$$

2) Bounds for Flat Dominant Valuations: Assume a flat dominant utility profile and let $x_i = u_{n-i}^2 - u_{n-i+1}^2$, for $i = 1, .., n-1$, and $x_n = u_1^1 - u_1^2$. (See Fig. 2.) We can then rewrite (3) as

$$s \cdot \sum_{i=s}^{n} x_i \geq r \cdot \sum_{i=r}^{n} x_i, \forall r \in \{1, \dots, n\}.$$

The difference in value between the efficient allocation and the sequential allocation (s, t) is

$$n \cdot u_1^1 - \left(s \cdot u_1^1 + \sum_{i=1}^{n-s} u_i^2 \right) = \sum_{i=s+1}^{n} (i - s) \cdot x_i.$$

Likewise, we have $\sum_{i=1}^{n} x_i = u_1^1 - u_n^2$, and so the efficiency loss can be written as

$$\frac{\sum_{i=s+1}^{n}(i-s) \cdot x_i}{n \cdot (\sum_{i=1}^{n} x_i + u_n^2)}.$$

The next lemma bounds this for a given allocation of j units to agent 1.

Lemma 5: The maximum efficiency loss for the sequential allocation $(j, n-j)$ assuming a flat dominant utility profile is $\frac{j}{n} \sum_{i=j}^{n-1} \frac{1}{i+1}$.

The proof of Lemma 5 is given in Appendix A. It follows that the worst case efficiency of the sequential auction with a flat dominant utility profile is

$$\eta'(n) = \min_{j \in [1,\dots,n]} \left\{ 1 - \frac{j}{n} \sum_{i=j}^{n-1} \frac{1}{i+1} \right\}, \tag{4}$$

which converges to $1 - e^{-1}$ as $n \to \infty$.

In Appendix B, we show that the flat dominant utility profile achieves the worst-case efficiency, so that $\eta'(n) = \eta(n)$. That completes the proof of Theorem 2.

3) Worst-Case Examples: Next we construct an example to show that Theorem 2 is asymptotically tight. Consider an auction with n goods and suppose that $u_1^1 = \dots = u_n^1 = 1$, and $u_1^2 = 1 - \frac{j}{n} + \varepsilon_1, u_2^2 = 1 - \frac{j}{n-1} + \varepsilon_2, u_3^2 = 1 - \frac{j}{n-2} + \varepsilon_3, \dots, u_{n-j}^2 = 1 - \frac{j}{j+1} + \varepsilon_{n-j}, u_{n-j+1}^2 = 0, \dots, u_n^2 = 0$, where $j \in \{1, \dots, n\}$ and $\varepsilon_i > 0$ for all i. From Corollary 4, it follows that if agent 1 receives j units, her terminal payoff is $j \cdot 1$. However, if agent 1 receives $j+1$ goods, her payoff becomes $(j+1)(1-u_{n-j}^2) = j-(j+1)\varepsilon_{n-j}$, which is smaller than j. Similarly, agent

1's payoff is smaller than j if she is allocated $r \neq j$ units. Therefore the sequential allocation is $(j, n-j)$. As the ε_i's approach zero, the efficiency of this outcome approaches

$$\frac{j + \sum_{i=1}^{n-j} u_i^2}{n} = 1 - \frac{j}{n} \sum_{i=j}^{n-1} \frac{1}{i+1}. \tag{5}$$

Minimizing (5) over $j \in [1, \ldots, n]$ gives the worst-case efficiency for this class of valuations. Comparing with (4) shows that these allocations give the worst-case efficiency for each n.

Table I shows the marginal values that give the lowest efficiency $\eta(n)$, which is also shown. As can be seen, $\eta(n)$ is decreasing with n. As $n \to \infty$, these quantities approach the bound from Theorem 2.

B. Efficiency Bound With Increasing/Decreasing Marginal Values

We now assume that agent 1 has increasing marginal values, while agent 2's marginals are decreasing. As noted previously, this may arise in the full-spread case due to interference.

Theorem 6: If the marginal values of one agent are decreasing and the other's are increasing, then $\eta(n) \leq \frac{1}{n}$.

Proof: Consider the following marginal values: $u_1^1 = a, u_2^1 = \cdots = u_n^1 = \varepsilon$ and $u_1^2 = \cdots = u_{n-1}^2 = 0, u_n^2 = b$ with $b > a + n\varepsilon$ and ε small. If the sequential auction reaches $(0, n-1)$, then agent 1 bids a for the last unit and agent 2 bids b. Hence, agent 2 wins the unit and pays a. The value of $(0, n-1)$ to agent 2 is therefore $b - a$. By backward induction, the value of $(0, 1)$ to agent 2 is $b - (n-1)\,a - \frac{(n-2)(n-1)}{2}\,\varepsilon$, assuming agent 2 wins all $n-1$ units after the first unit. Similarly, the value of $(1, 0)$ to agent 1 is $a + (n-1)\,\varepsilon$. The sequential outcome is inefficient if agent 2's value of $(0, 1)$ is less than agent 1's value of $(1, 0)$, i.e., if $b < n\,a + \frac{(n-1)n}{2}\,\varepsilon$. In that case, the efficiency of the sequential auction outcome is given by

$$\frac{a + (n-1)\,\varepsilon}{b} > \frac{a + (n-1)\,\varepsilon}{n\,a + \frac{(n-1)n}{2}\,\varepsilon}.$$

Letting $a \to \infty$ and/or $\varepsilon \to 0$, the efficiency approaches $\frac{1}{n}$. ∎

This theorem shows that when one agent has an increasing marginal the worst-case efficiency can go to zero as the number of goods increases. From Theorem 2, this is not the case when each agent's marginals are decreasing.

C. Efficiency with Constrained Marginal Values

As indicated in the preceding sections, the marginal values that achieve the worst-case efficiency in each case are quite special. With additional constraints on the marginal values, we expect the worst-case efficiency to increase. Here we illustrate this for $n = 2$ goods.

First, we consider decreasing marginal valuations for both agents, and assume that $u_2^1 = \lambda_1 u_1^1$ and $u_2^2 = \lambda_2 u_1^2$, where $\lambda_1 < 1$ and $\lambda_2 < 1$. In this case, it can be shown that the sequential allocation is not efficient if and only if $u_1^1 > \lambda_1 u_1^1 > u_1^2 > \lambda_2 u_1^2$ or $u_1^2 > \lambda_2 u_1^2 > u_1^1 > \lambda_1 u_1^1$. The worst-case efficiency with these constrained marginal valuations is given by

$$\eta(2; \lambda_1, \lambda_2) = \frac{2 + \lambda_1 - \lambda_2}{(1 + \lambda_1) \cdot (2 - \lambda_2)}, \tag{6}$$

where $\frac{u_1^2}{u_1^1} < \lambda_1 < 1$ and $0 < \lambda_2 < 1$. Note that $\eta(2; \lambda_1, \lambda_2) \geq 3/4$, which is equal to the bound from (4), i.e. restricting the marginals in this way decreases the efficiency loss. As $\lambda_1 \to 1$ and $\lambda_2 \to 0$, this bound holds with equality.

For the case in which one agent has decreasing marginal values and the other has increasing marginal values, we let $u_2^1 = \lambda_1 u_1^1$, and $u_2^2 = \lambda_2 u_1^2$, where $\lambda_1 < 1$ and $\lambda_2 > 1$. In this case, any ordering of marginal values can lead to an inefficient allocation. Hence all orderings must be considered to compute the worst-case efficiency. As an example, assume that $\lambda_2 u_1^2 > u_1^2 > u_1^1 > \lambda_1 u_1^1$. Then the worst-case efficiency is given by

$$\eta(2; \lambda_1, \lambda_2) = \frac{2 - \lambda_1 + \lambda_2}{(2 - \lambda_1) \cdot (1 + \lambda_2)}, \tag{7}$$

where $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 2$. Here we have $\eta(2; \lambda_1, \lambda_2) \geq 2/3$, and equality holds as $\lambda_1 \to 0$ and $\lambda_2 \to 2$. Again, restricting the marginal values increases the worst-case efficiency.

V. Simulation Results

In this section we present simulation results for two-user bandwidth and power auctions. For these results we randomly place two transmitters and receivers within a given region, as illustrated in Fig. 3. Specifically, user 1's transmitter is uniformly placed within a circle of radius $d_0 = 50\,m$ centered at user 2's receiver. This captures the scenario in which a user experiences a single dominant interferer. User 1's receiver is then randomly placed within a circle of radius d_0 centered at user 1's transmitter, and similarly, user 2's transmitter is randomly placed within a circle of radius d_0 centered at user 2's receiver. Given these locations, we set each channel gain $h_{ij} = l_{ij}^{-4}$ where l_{ij} is the distance between transmitter i and receiver j. For a given allocation a user's utility is assumed to be the rate given by (1) or (2), with $W = 25$ MHz, and $N_0 = -174$ dBm/Hz. In the bandwidth auction, W is divided into n units of W/n Hz and both users transmit using power $P_i = P_{max} = 10^{-6}$ watts. In the power auction, we assume that $P_2 = P_{max}$ and $P_1 = n_1 P_{max}/n$, where again $P_{max} = 10^{-6}$ watts. Both users spread over the entire bandwidth W.

A. Bandwidth Auction

We first show results for the bandwidth auction with $n = 2$ units. We define the *worst possible efficiency* for a given realization as the ratio of minimum sum utility to maximum sum utility over the three possible bandwidth allocations. Figure 4 shows the empirical probability distribution function (PDF) for the worst possible efficiency over 10^4 simulation runs. This shows that without an appropriate resource allocation mechanism, the efficiency can be very low.

Figure 5 shows the empirical cumulative distribution function (CDF) of the efficiency of the sequential equilibrium. Curves are shown for different values of n. For $n = 2$ this figure shows a substantial improvement in efficiency relative to the worst possible allocation in Figure 4. For $n = 2$, the lowest efficiency is 0.844, and the auction achieves an efficient allocation for more than 80% of the realizations. The lowest efficiency is significantly higher than the worst-case efficiency of $3/4$ given in Section IV-A3. This is due to the nature of the rate utility function, which constrains the possible marginal values as in Section IV-C. Here, each agent i's utility function has the form $U_i(s) = s(W/2)\log_2(1 + \frac{2\beta_i}{s})$, where $\beta_i = \frac{|h_{ii}|^2 P_i}{N_0 W}$. For the parameters used in the simulation it follows that $\beta_i \in [1.6, \infty)$. The resulting marginals satisfy the constraints in Sect. IV-A3, with $\lambda_i \in [.45, 1]$. From (6), the worst-case efficiency occurs when $\lambda_1 = 1$ and $\lambda_2 = 0.45$, which gives $\eta(2; 1, 0.45) = 0.82$, only slightly less than the observed lowest efficiency.

As n increases, Figure 5 shows that the smallest observed efficiency increases from 0.844 when $n = 2$ to 0.914 when $n = 20$. This is in contrast to the results in Section IV-A3, which show that the worst-case efficiency decreases with n. The observed increase is due to the fact that as n increases, the specific marginal values, which achieve the worst-case efficiency, are much less likely to occur. However, the fraction of realizations for which the full efficiency is achieved decreases as n increases. In part, this is simply due to the increase in number of possible outcomes (allocations) with n.

B. Power Auction

Figure 6 shows the PDF of the worst possible efficiency for the power auction with $n = 2$ units. Figure 7 shows the CDF of the efficiency of the sequential allocation for different values of n. Unlike the bandwidth auction, the smallest efficiencies observed in the simulations are close to $1/n$, as predicted by Theorem 6. (For example, with $n = 2$ the smallest observed efficiency is 0.575.) Because of the interference, the marginal value of the second unit for agent 2 can be very large relative to the marginal value of the first unit, which leads to the worst-case efficiency. For $n = 2$ the sequential power auction still achieves the efficient allocation for more than 85% of the realizations. This fraction decreases as n increases.

Finally, we remark that our results for both the power and bandwidth auctions only indicate efficiency loss relative to the maximum utility for that mechanism. Further results comparing the efficiency across mechanisms show that in addition to having lower efficiency loss, the bandwidth auction typically achieves a higher sum utility than the power auction.

VI. Sequential second price auction for three or more agents

Next we turn to the case where $k > 2$ agents are participating in the bandwidth auction.[14] The main question we address is whether or not the auction has an equilibrium.[15] In a single unit second-price auction, existence of an equilibrium follows from the uniqueness of dominant strategies for all agents. From Theorem 1, a similar argument applies for a two agent sequential auction, namely there is a unique dominant subgame perfect strategy for each agent. However, with $k > 2$ agents, we will show by example that one or more agents may not have a unique dominant strategy. Hence, it is plausible that there exist no equilibria (as in first price auctions with full information) or a multiplicity of pure and mixed strategy equilibria. Our main result is to show the existence of at least one pure strategy equilibrium.

[14] As discussed in Section II, due to the interdependence of the utility functions, the power auction with more than two agents is not considered.

[15] Note that this game has infinite strategy spaces and discontinuous pay-off functions, hence classical equilibria existence theorems may not apply.

Consider sophisticated bidding for $k > 2$ agents in an n-unit auction. As in the two agent case, the last round of the auction is identical to a standard second price auction for the nth good, and so it is a dominant strategy for all agents to bid their valuations. Given full information, all agents again know the allocations and payments on the last round. Hence, we can think of the penultimate round as a second price auction over the right to participate in one of k possible auctions in the last round whose valuations are known.

In the two agent auction at each node the choice is between two possible sub-auctions. An agent's value for one sub-auction over the other is captured by the difference in payoffs between them. Since any sub-auction has a unique equilibrium path, the valuations of the sub-auctions and hence the sophisticated bids are well defined. With $k > 2$ agents, even in the penultimate auction, the choice may be between k alternative second price auctions for which some or all of the agents have different payoffs. Each agent may then have several non-dominated strategies, and as the next example shows, there may be multiple sub-game perfect equilibria. If the equilibrium is not unique, the valuation of the penultimate round may depend on the choice of equilibrium. The same applies, of course, to any node further up the game tree. We therefore define a sophisticated bidding strategy as a strategy that, for each node of the game tree, maximizes the agent's payoff over final outcomes for a given equilibrium strategy on each of the subtrees. In other words, an agent chooses a sophisticated strategy that subsumes some choice of equilibria on the subtrees and maximizes expected payoff for the corresponding valuations.

A. Example

Figure 8 shows an example of a sequential second price auction with three agents and three units which has multiple inefficient equilibria. The marginal valuations of all three units are 10 for agent 1 and 9, 1 and 0 for agents 2 and 3. Since the last round is a second price auction, bidding marginal values is a equilibrium on each of the final round subtrees. Using these values, in the subtree of the penultimate round corresponding to the allocation of the first unit to agent 1 we get the values $[21, 0, 0]$, $[11, 9, 0]$ and $[11, 0, 9]$ of the second unit being allocated to agents 1, 2 and 3, respectively.[16] This implies that bidding 10 for agent 1 and 9 for agents 2 and 3 are dominant strategies. The value of this subtree is therefore $[12, 0, 0]$. In the penultimate

[16]Here each component denotes the corresponding value for that agent.

round corresponding to the allocation of the first unit to agent 2 we get the values $[11, 9, 0]$, $[1, 9, 10]$ and $[9, 9, 9]$; hence, the dominant bids are 2, 9, 1 (since agent 1 knows that agent 2 loses regardless of agent 1's bid) and the valuation is $[9, 9, 7]$. By symmetry, the valuation of the third penultimate subtree is $[9, 7, 9]$.

Turning to the first round, it follows that agent 1 has a dominant bid of 3 while the two other agents have a choice between 2 and 9. In this case agents 2 and 3 must coordinate to avoid simultaneously bidding high or low thus the pure strategy equilibria bids for this round are 3, 2, 9 and 3, 9, 2. There also exists a mixed strategy where both agents 2 and 3 flip a fair coin and decide between 2 and 9.

B. Existence

To show that there exists at least one equilibrium with $k > 2$ agents, we define a *second price bidding* mechanism which is a generalization of a second price auction.

Definition 2: A k-second price bidding mechanism is a k-agent mechanism with action profiles in $\mathbb{R}^k_+$ and a finite outcome set $\{A_1, \ldots, A_k\}$ where the valuation of agent i for A_j is $a^i_j \in \mathbb{R}$ and $a^j_j \geq a^j_i$ for any $i \neq j$. The outcome as a function of the actions $(b_1, \ldots, b_k)$ is given by $\nu(b_1, \ldots, b_k) = A_i$ when $b_i = \max_j b_j$ and the payment in this case is $p_i(b_1, \ldots, b_k) = \max_{j \neq i} b_j$ and $p_j(b_1, \ldots, b_k) = 0$ for $j \neq i$.

It is easy to see that this reduces to a second price auction if $a^j_i = 0$ for $j \neq i$.

Lemma 7: A second price bidding mechanism has at least one pure strategy equilibrium that survives iterated elimination of dominant strategies.

Proof: For each agent i let $B_i = \{a^i_i - a^i_j : \ j \neq i\}$, the set of value differences between outcomes, $\beta_i = \min B_i$. Without loss of generality, $b_1 = a^1_1 - a^1_2 = \max \cup_i B_i$, namely the largest valuation gap is between the valuations of agent 1 for the outcomes A_1 and A_2.

We show that if $b_2 = \max B_2 > \max_{i>2} \beta_i$ then the bidding profile $(b_1, b_2, \beta_3, \ldots, \beta_k)$ is an equilibrium. With this profile the outcome is A_1 with $p_1(b_1, b_2, \beta_3, \ldots, \beta_k) = b_2$ and $p_i(b_1, b_2, \beta_3, \ldots, \beta_k) = 0$ for $i > 1$. Agent 1's payoff is then $a^1_1 - b_2$. The only deviation of agent 1 that would change the outcome is to bid below b_2 which, by the assumption on b_2, would give the outcome A_2. Agent 1's payoff in this case is a^1_2 and the difference is $a^1_2 - a^1_1 + b_2 = b_2 - b_1 < 0$ from the maximality of b_1. If agent $i > 1$ bids above b_1, then her payoff is $a^i_i - b_1$ compared to

a_1^i at A_1, hence by deviating she would gain $a_i^i - b_1 - a_1^i$ which, again by the maximality of b_1, is negative. Thus, no agent can make a positive gain from deviating.

If $b_2 = \max B_2 < \max_{i>2} \beta_i$ then w.l.o.g. $\beta_3 > b_2$. By induction there exists a pure strategy equilibrium for the $k-1$ bidding mechanism derived from excluding agent 2. The base of the induction follows since for two agents trivially it must be that $b_2 > \max_{i>2} \beta_i$.

Since we are removing one of the agents in the new game, the new sets of value differences are subsets of the previous B_i's. This implies that their minimal elements β_i' satisfy $\beta_i' \geq \beta_i$, and therefore agent 3 bids above b_2. If agent 3 is not the highest bidder in the new game, or if at least two agents are bidding above b_2, then taking the equilibria bids in the new game and letting agent 2 bid b_2 would give the same allocation and payments as in the k-agent game. Since b_2 is the maximal gain agent 2 could obtain from changing the outcome, she has no incentive to bid above b_2. Any profitable deviation for the other agents would be a profitable deviation in the new game contradicting the choice of bids as an equilibrium. If agent 3 is the highest bidder in the new game and the second highest bid is below b_2 then the same argument shows that adding agent 2 to the equilibrium profile in the new game would not change the bidding incentives of the agents apart from agent 3. Since $a_3^3 - a_3^2 > \beta_3 > b_2$, it follows that agent 3 has no incentive to deviate either. Thus we get a pure strategy equilibrium for the k agent mechanism. These strategies are not dominated hence this equilibrium survives iterated elimination of dominant strategies.

If $\max B_2 = \max_{i>2} \beta_i$ then the outcome depends on the tie breaking rule used in the auction. For any reasonable rule, such as random choice, a pure strategy equilibrium can be constructed in a similar manner. ■

Theorem 8: The multi-agent sequential second price auction has a pure strategy equilibrium.

Proof: An induction on the depth of the game tree of a sequential second price auction shows that each round of the sequential auction is strategically equivalent to a second price bidding mechanism where the valuations of the subtrees depend on the choice of equilibrium outcome of the bidding mechanism on the subtree. ■

C. *Efficiency loss*

We conclude this section with a few comments about the efficiency for $k > 2$ agents. First we note that the worst-case efficiency will not increase as the number of users increase. This follows from the fact that we can always select the marginals of the additional users to be arbitrarily small. In fact for $n = 2$ goods and an arbitrary number of users with decreasing marginals it can be shown that the worst-case efficiency is exactly the same as in the $k = 2$ case (i.e. it is $3/4$). Figure 9 shows simulation results for the bandwidth auction with $k = 3$ agents and $n = 2$ and $n = 5$ goods. The parameters are the same as those in Section V. For comparison the results to $k = 2$ agents are also shown. It can be seen that the efficiency with $k = 3$ agents is stochastically larger than that with $k = 2$ agents for both $n = 2$ and $n = 5$. A likely explanation for this is that with randomly placed agents the probability of a "bad" choice of utilities arising decreases as the number of agents increase.

VII. Conclusions

We have considered a sequential second price auction for allocating n units of bandwidth or power among non-cooperative wireless devices. This mechanism is relatively simple and requires little information exchange among users, which may make it attractive for dynamic bandwidth or power allocation among secondary users who wish to share spectrum with the primary user (spectrum owner or licensee). Our main analytical results characterize the worst-case efficiency of the subgame perfect equilibrium for two users with full knowledge of bidding histories and user utilities. For a bandwidth auction (decreasing marginal utilities), the worst-case efficiency decreases with n and converges to $1-e^{-1}$. For the power auction, where one user has decreasing marginal utilities and the other has increasing values, the worst-case efficiency bound is no greater than $1/n$.

Although the worst-case efficiency loss due to sophisticated bidding can be significant, simulation results with randomly placed users show that with the rate utility function, the sequential auction typically gives the efficient allocation. Furthermore, when the equilibrium is inefficient, the efficiency loss is typically less than the worst-case efficiency loss. This is due to the rate utility function, which places constraints on the ratios of marginal utilities for the successive units being auctioned.

For more than two users, we show that the sequential second price auction still has a pure strategy equilibrium. In this case, however, the equilibrium may not be unique and so some coordination of the users may be needed to decide on a particular outcome. Assuming a particular equilibrium, simulation results show that for the bandwidth auction the efficiency typically improves when the number of agents increases from 2 to 3. Completely characterizing the efficiency with an arbitrary number of goods and agents is an open problem.

In the absence of full information about other users' utilities, each user may attempt to strategize bidding by assuming a distribution over those utilities. Computing equilibria and efficiency loss in that case is another open problem, although in general less information seems more likely to encourage bidding according to marginal utilities, which leads to an efficient allocation in the bandwidth auction. Extensions to joint power and bandwidth auctions are also interesting possibilities for future work.

APPENDIX A
PROOF OF LEMMA 5

Let $x_1, \dots, x_n$ be the solution to the following linear program:

$$\max_{x_1,\dots,x_n \geq 0} \quad \phi(x_1, \dots, x_n) := \sum_{i=j+1}^{n} (i-j) \cdot x_i, \tag{8}$$

$$\text{subject to:} \quad j \cdot \sum_{i=j}^{n} x_i \geq r \cdot \sum_{i=r}^{n} x_i, \quad \forall r \neq j, \tag{9}$$

$$\sum_{i=1}^{n} x_i = u_1^1 - u_n^2. \tag{10}$$

From the discussion preceding Lemma 5, the maximum efficiency loss for the sequential allocation $(j, n-j)$, assuming a flat dominant utility profile, is

$$\max_{u_1^1, u_n^2 : u_1^1 > u_n^2 \geq 0} \frac{\phi(x_1, \dots, x_n)}{n(\sum_{i=1}^{n} x_i + u_n^2)}. \tag{11}$$

To complete the proof, we will show that the solution to this optimization has the desired form.

First note that the linear program only depends on $u_1^1 - u_n^2$, and (11) is decreasing in u_n^2. Hence we can always increase the efficiency loss by setting $u_n^2 = 0$. In addition, because the objective function only depends on $x_{j+1}, \cdots, x_n$, we set $x_1 = \cdots = x_{j-1} = 0$ to make the largest feasible region. With this choice of x_i, the only constraints in (9), which can be binding, are those for $r > j$. It is easy to see that at optimality, the remaining constraints are binding. Therefore, the constraints (9) and (10) can be written as the following linear system:

$$
\begin{aligned}
x_j + x_{j+1} + x_{j+2} + \ldots + x_n &= u_1^1 \\
x_{j+1} + x_{j+2} + \ldots + x_n &= \frac{j}{j+1} u_1^1 \\
x_{j+2} + \ldots + x_n &= \frac{j}{j+2} u_1^1 \\
&\vdots \\
x_n &= \frac{j}{n} u_1^1.
\end{aligned}
$$

This set of equations gives the following unique feasible solution.

$$
x_i = \begin{cases} \frac{j}{i(i+1)} & i = j, \ldots, n-1, \\ \frac{j}{n} & i = n. \end{cases}
$$

Hence from (11), the maximum efficiency loss is $\frac{j}{n} \sum_{i=j}^{n-1} \frac{1}{i+1}$. ■

APPENDIX B
PROOF OF THEOREM 2

Suppose that $u_1^1 \geq \ldots \geq u_n^1$ and $u_1^2 \geq \ldots u_n^2$, and let $(l, n-l)$ denote the efficient allocation. After auctioning m $(\leq n)$ units, the sequential game reaches a decision node where either agent 1 or agent 2 obtains her efficient allocation (l for agent 1 or $n-l$ for agent 2). For that agent the marginal values of the remaining units must be smaller than that for the other agent. (See Figure 10.) Up to this decision node, there is no loss in efficiency. Any efficiency loss in the final allocation procures in the subgame tree rooted at this decision node. Therefore, the efficiency loss of the full game tree cannot be larger than the efficiency loss of this subgame tree. Since the utility profile associated with the subgame tree is dominant, the worst-case efficiency must always correspond to a dominant utility profile.

We now show that changing a dominant utility profile to a flat dominant profile can only decrease efficiency. Given a dominant utility profile, from Corollary 4, if we replace the marginals of the first agent with $\bar{u}_1^1 = \ldots = \bar{u}_n^1 = u_n^1$, then we must have

$$
s \cdot (\bar{u}_1^1 - u_{n-s+1}^2) \geq r \cdot (\bar{u}_1^1 - u_{n-r+1}^2).
$$

for any $r \neq s$. This implies that for the flat dominant profile $\bar{u}_1^1 = \ldots = \bar{u}_n^1 \geq u_1^2 \geq \ldots \geq u_n^2$, the sequential equilibrium $(\bar{s}, \bar{t})$ satisfies $\bar{s} \leq s$. Hence this change in utility profile can only decrease efficiency. ■

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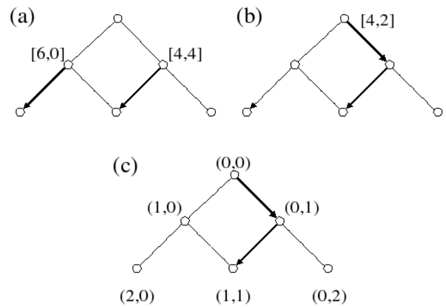

Fig. 1. Example of the sequential auction with $k = 2$ agents and $n = 2$ resource units. (a) and (b) show the valuations of each node and (c) shows the equilibrium path.

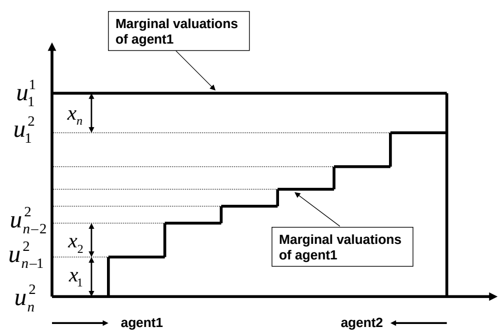

Fig. 2. The extremal case: Agent 1 has constant marginal values and the marginals of agent 2 are below the marginals of agent 1. The axis for agent 1 is from left to right and the axis for agent 2 is from right to left.

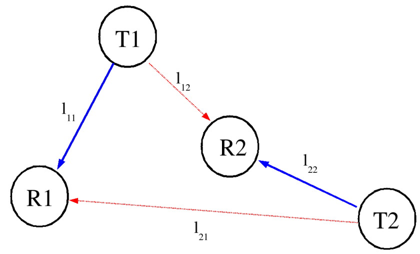

Fig. 3. Simulation scenario in which the location of the transmitter T1 is uniformly distributed within a circle centered at R2, and R1 and T2 are placed at random locations within circles centered at T1 and R2, respectively. ($l_{11} \leq d_0$, $l_{22} \leq d_0$ and $l_{12} \leq d_0$)

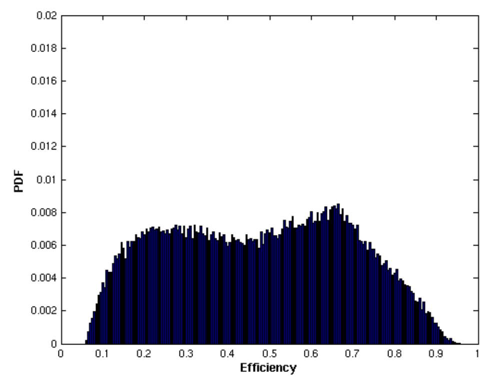

Fig. 4. Empirical PDF of the worst possible efficiency of the sequential bandwidth auction with two agents and $n = 2$ units.

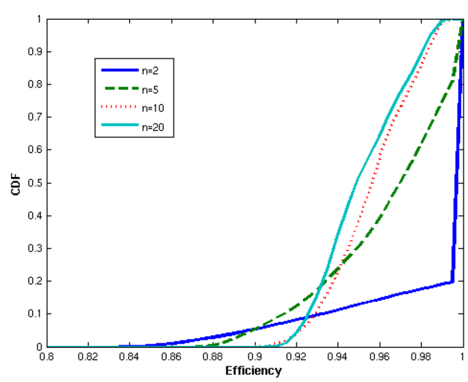

Fig. 5. Empirical CDFs of the efficiency of the two-user sequential allocation for the bandwidth auction with different n. The transmitted power $P = 10^{-6}$ and $d_0 = 50$ m.

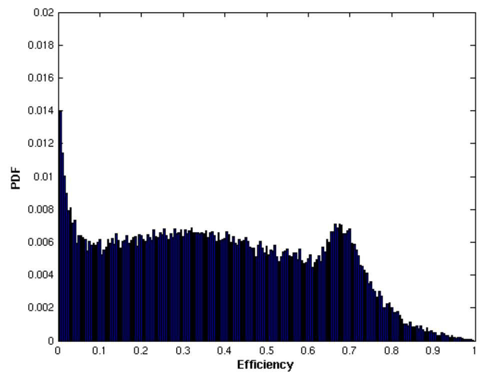

Fig. 6. Empirical PDF of the worst possible efficiency for the power auction with two users and $n = 2$ units.

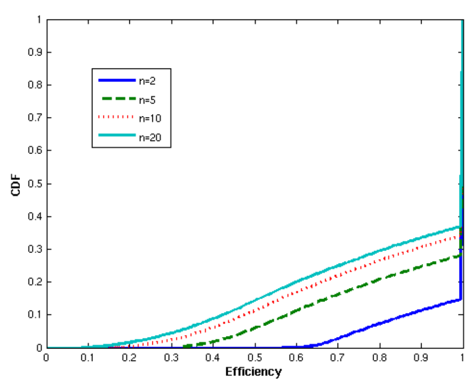

Fig. 7. Empirical CDF of the efficiency of the sequential equilibrium for the two-user power auction with different values of n.

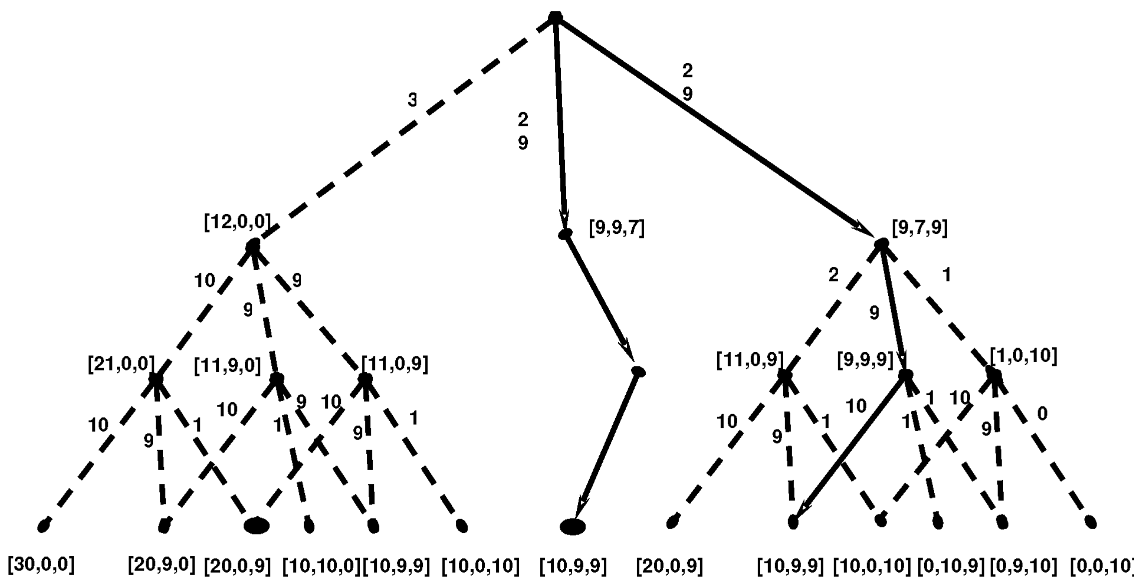

Fig. 8. Example of a directed graph G representing an auction with $n = 3$ units and $k = 3$ agents. Each node on the graph is labeled with the valuation of the subtree rooted at that node. The edges are labeled with the value of the resource unit to the corresponding agent. The two solid paths correspond to the two pure strategy equilibria.

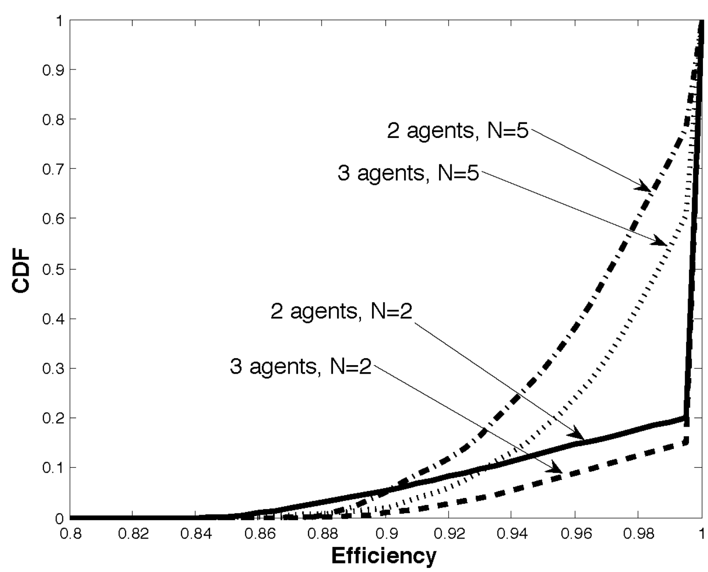

Fig. 9. Empirical CDFs of the efficiency of the sequential allocation for the bandwidth auction with $k = 2$ and $k = 3$ users and $n = 2$ and $n = 5$ goods.

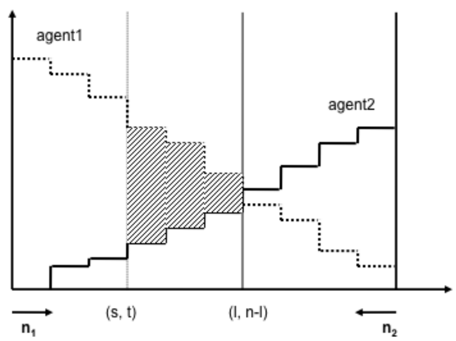

Fig. 10. Marginal values of two agents. n_1 (n_2) is the number of units that agent 1 (2) obtains along the sequential auction. $(k, n-k)$ is the optimal allocation and (s, t) is the sequential allocation. The shadowed region shows the efficiency loss.

n	Marginals	j^*	$\eta(n)$
2	$1, 1\ \ ;\ \ 1/2 + \varepsilon_1, 0$	1	$3/4$
3	$1, 1, 1\ \ ;\ \ 2/3 + \varepsilon_1, 1/2 + \varepsilon_2, 0$	1	$13/18$
4	$1, 1, 1, 1\ ;\ 1/2 + \varepsilon_1; 1/3 + \varepsilon_2, 0, 0$	2	$17/24$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞			1- $\frac{1}{e}$

TABLE I
MARGINAL VALUES AND CORRESPONDING WORST-CASE EFFICIENCY ACHIEVED BY TWO USER THE SEQUENTIAL AUCTION FOR GIVEN n.