

# Optimal Spectrum Allocation in Gaussian Interference Networks

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**Abstract**—We consider a wireless network in which multiple users share a common spectrum band, modeled as a Gaussian interference network. Our objective is to choose the transmit power spectral density of each user to maximize the sum rate, subject to individual power constraints at each transmitter. This includes frequency-orthogonal transmissions and full-spreading with power control as special cases. Assuming single-user receivers and treating interference as noise, the resulting optimization problem is non-convex. Nevertheless, for a two-user model with flat-fading we completely characterize the optimal solution. In particular, we show that the optimal power allocation consists of each user using a piecewise constant power allocation over at most 3 frequency bands, where at most one band is shared by the two users. Extensions to more than two users and a distributed algorithm to implement the optimal allocation are also presented.

## I. INTRODUCTION

Managing interference is critical to enable multiple transmitters to operate in a common frequency band. This is becoming more challenging as wireless devices proliferate and are deployed in a more ad hoc manner (for example in unlicensed bands). Judiciously allocating the power spectral density of each transmitter is one of the key techniques for mitigating interference in such settings. Indeed, most common spectrum sharing approaches can be viewed as special types of spectrum allocation, including frequency division multiplexing (FDM) and full spreading (FS) with power control.

In this paper, we consider the optimal allocation of power spectral densities to maximize the sum-rate of a set of disjoint transmitter-receiver pairs operating in a common spectrum band. We focus on a continuous formulation of this problem, in which each user can choose an arbitrary power spectrum across frequency. The underlying channel is modeled as a Gaussian interference network with flat fading and each user is assumed to use Gaussian signaling and treat interference as noise<sup>1</sup>, so that each user's rate depends on the received signal-to-interference plus noise (SINR) ratio across the spectrum band. The resulting problem is non-convex; however, for two-users in a flat-fading environment we are able to completely characterize the optimal power allocation. In particular, we show that each user will have a piecewise constant power

allocation over at most three frequency bands and that the two users will share at most one of these bands.

Our approach is based in part on extending a result in [4], which shows that in an  $M$  user network with flat fading, each user employs a frequency flat power allocation over at most  $2M$  sub-bands. We refine this to show that in fact  $M+1$  bands are sufficient and then use this to reformulate the problem as one of finding a convex combination of a finite set of signal-to-noise ratios (SNRs) to optimize a corresponding convex combination of a sum-rate per unit bandwidth function. We briefly discuss the extension of this to more than two users and also present a distributed algorithm, where users alternately update bandwidth and power allocations to approach the optimal solution. This algorithm is based in part on work in [5], [6], which studies distributed power allocation for OFDMA-based networks.

In related work, spectrum management has been studied for OFDMA-based wireless networks as well as for wire-line digital subscriber line (DSL) networks. For example, in [7], the iterative water-filling algorithm is proposed for distributed spectrum management in Gaussian interference channels. In general this algorithm may not converge or may converge to a sub-optimal solution. In [8]–[10] properties of centralized spectrum optimization are studied using dual-based approaches. It is shown that in certain cases there is zero duality gap despite the non-convexity of the problem. Also [10] shows that many instances of this problem are NP-hard. Much of this work applies to the case where each channel exhibits frequency selective fading. In such a setting, in addition to mitigating interference, spectrum allocation is used to exploit frequency diversity. Here, to focus on the frequency sharing aspects of spectrum allocation, we assume frequency flat fading.

The paper is organized as follows. We first formulate the optimization problem for  $M$  users in terms of each user's continuous power spectral density and then give a reformulation in terms of a finite number of SNRs. Next, we turn to the case of  $M = 2$  users and characterize the complete solution. Extensions to the multiple-user case are briefly discussed, followed by the alternating bandwidth and power algorithm. Technical proofs are omitted due to page limitation; the reader is referred to the extended version of this paper for more details.

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<sup>1</sup>From an information theoretic perspective, treating interference as noise may not be optimal except in the case of very weak interference (see e.g. [1]–[3]); however, this is a common assumption in current practice.

## II. PROBLEM FORMULATION

Consider an  $M$ -user Gaussian interference channel with flat fading. Without loss of generality we assume that the available spectrum corresponds to the frequency-band  $[0, B]$ . The set of users is denoted by  $\mathcal{M} = \{1, \dots, M\}$ , where each user indicates a distinct transmitter/receiver pair. For each  $i \in \mathcal{M}$ , there is associated a transmit power spectral density  $p_i : [0, b] \mapsto \mathbb{R}^+$ , which must satisfy the power constraint  $\int_0^B p_i(f) df \leq P_i$ . For a given choice of power spectra for each user  $j$ , each user  $i$  receives a rate given by <sup>2</sup>

$$R_i = \int_0^B \log \left( 1 + \frac{h_{ii} p_i(f)}{N_0 + \sum_{j \neq i} h_{ji} p_j(f)} \right) df,$$

where  $h_{ik}$  denotes the direct channel gain from the transmitter of user  $i$  to the receiver of user  $j$  and  $N_0$  denotes the noise power spectral density.

Our objective is to determine the power spectral densities which maximize the sum-rate across all users. This is given by the following optimization problem:

$$\begin{aligned} \max_{\{p_i(f)\}_{i \in \mathcal{M}}} & \sum_{i \in \mathcal{M}} \int_0^B \log \left( 1 + \frac{h_{ii} p_i(f)}{N_0 + \sum_{j \neq i} h_{ji} p_j(f)} \right) df, \\ \text{s.t.} & \int_0^B p_i(f) \leq P_i, \quad p_i(f) \geq 0, \quad i \in \mathcal{M}. \end{aligned} \quad (1)$$

It was proved in [4] that any point in the achievable rate region for this model can be obtained with power allocations that are piece-wise constant in at most  $2M$  disjoint frequency sub-bands in  $[0, B]$ . Using a similar argument, this can be refined to yield the following lemma which characterizes the optimal power allocation for (1).

**Lemma 1.** *There exists an optimal solution to (1) in which each user employs a piece-wise constant power allocation in at most  $M + 1$  disjoint frequency sub-bands in  $[0, B]$ .*

Using Lemma 1, the optimization in (1) can be reformulated as optimizing over the bandwidth  $B_k$  and each user  $i$ 's power allocation  $p_i^k$  for each of  $M + 1$  sub-bands ( $k \in \mathcal{K} = \{1, \dots, M + 1\}$ ).<sup>3</sup> This results in the following optimization

$$\begin{aligned} \max & \sum_{k \in \mathcal{K}} B_k \sum_{i \in \mathcal{M}} \log \left( 1 + \frac{h_{ii} p_i^k}{N_0 B_k + \sum_{j \neq i} h_{ji} p_j^k} \right), \quad (2) \\ \text{s.t.} & \sum_{k \in \mathcal{K}} B_k = B; \quad B_k \geq 0, \quad k \in \mathcal{K}; \\ & \sum_{k \in \mathcal{K}} p_i^k \leq P_i, \quad p_i^k \geq 0, \quad i \in \mathcal{M}, \quad k \in \mathcal{K}. \end{aligned}$$

Furthermore, if we define  $\alpha_k = B_k/B$ , which is the fraction of the total bandwidth that is allocated to the  $k$ th sub-band, and  $x_i^k = p_i^k/(N_0 B_k)$ , which is the SNR of user  $i$  in sub-band

<sup>2</sup>For ease of analysis, we assume that all logarithms are natural logarithms in the following.

<sup>3</sup>Note if fewer than  $M + 1$  bands are used, then some of the  $B_k$ 's will be zero and by continuity we assume that the corresponding rate on those bands is also zero.

$k$ , then (2) can be rewritten as optimizing over  $\{\alpha_k, \mathbf{x}^k\}_{k \in \mathcal{K}}$ , where  $\mathbf{x}^k = (x_1^k, \dots, x_M^k)$ , i.e.,

$$\begin{aligned} \max & B \sum_{k \in \mathcal{K}} \alpha_k F(\mathbf{x}^k), \quad (3) \\ \text{s.t.} & \sum_{k \in \mathcal{K}} \alpha_k = 1; \quad \alpha_k \geq 0, \quad k \in \mathcal{K}; \\ & \sum_{k \in \mathcal{K}} \alpha_k x_i^k \leq \frac{P_i}{N_0 B}, \quad x_i^k \geq 0, \quad i \in \mathcal{M}, \quad k \in \mathcal{K}. \end{aligned}$$

Here,  $F$  indicates the sum-rate per unit bandwidth over a given band and is given by

$$F(\mathbf{x}) = \sum_{i \in \mathcal{M}} \log \left( 1 + \frac{h_{ii} x_i}{1 + \sum_{j \neq i} h_{ji} x_j} \right). \quad (4)$$

This can be interpreted as saying that the optimal spectrum allocation is given by finding a convex combination of the SNR vectors on each band  $k$ , which satisfies each user's power constraint and maximizes the corresponding convex combination of the sum-rate per unit bandwidth on each band  $k$  (given by  $F(\mathbf{x}^k)$ ).

## III. TWO USER MODEL

Now, we we turn to the two-user case, in which case  $\mathcal{K} = \{1, 2, 3\}$ . In this case, for each  $k \in \mathcal{K}$ , we simply write the SNR of user 1 as  $x_k = p_1^k/(N_0 B_k)$ , the SNR of user 2 as  $y_k = p_2^k/(N_0 B_k)$ , and replace (4) with

$$F(x, y) = \log \left( 1 + \frac{h_{11} x}{1 + h_{21} y} \right) + \log \left( 1 + \frac{h_{22} y}{1 + h_{12} x} \right).$$

Then, (3) becomes

$$\begin{aligned} \max & B \sum_{k \in \mathcal{K}} \alpha_k F(x_k, y_k), \quad (5) \\ \text{s.t.} & \sum_{k \in \mathcal{K}} \alpha_k = 1; \quad \alpha_k \geq 0, \quad k \in \mathcal{K}; \\ & \sum_{k \in \mathcal{K}} \alpha_k x_k \leq \frac{P_1}{N_0 B}, \quad x_k \geq 0, \quad k \in \mathcal{K}; \\ & \sum_{k \in \mathcal{K}} \alpha_k y_k \leq \frac{P_2}{N_0 B}, \quad y_k \geq 0, \quad k \in \mathcal{K}. \end{aligned}$$

### A. Main Result

In the following, we completely characterize the optimal solution to (5). Specifically, we show that for a given set of fixed channel gains, there are five possible scenarios for the optimal spectrum allocation, which depend on the power constraints of each user. In each scenario, both users transmit at their full power. This illustrated in Fig. 1, which shows an example of the range of power constraints that correspond to each scenario (indicated by the numbers 1-5). When the power constraints are large, as represented by scenario 1, it is optimal to divide the total bandwidth into two sub-bands, with one sub-band for each user's exclusive use (i.e. to employ FDM). On the other hand, for small power constraints, as represented by scenario 2, full-spreading (FS) at full powers for both users is optimal.

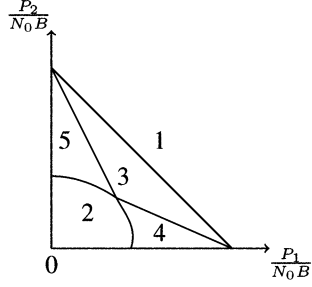


Fig. 1. Example of the scenarios for the optimal spectrum allocation.

The more interesting cases are the other three scenarios, with moderate power constraints. In scenario 3, there are three sub-bands: one shared by both users and one used exclusively by each. For scenarios 4 and 5, the optimal solution has two sub-bands; one shared by both users and one used exclusively by the user with the larger power constraint. The boundaries separating these scenarios are functions of the channel gains. We explicitly derive the boundaries for region 3; the remaining boundaries we can numerically determine.

### B. Optimal Sum-Rate: Tight Power Constraints and Optimality Criteria

Given fixed channel gain parameters, we define the optimal sum-rate as a function of the power constraints and derive some properties of this function. Note that the analysis in this section holds for  $M > 2$  users as well.

First, it can be shown that the power constraints must be tight for all users at the optimal solution to (3). Thus, (3) can be rewritten as

$$\begin{aligned} \max B \sum_{k \in \mathcal{K}} \alpha_k F(\mathbf{x}^k), \\ \text{s.t. } \sum_{k \in \mathcal{K}} \alpha_k = 1; \quad \alpha_k > 0, \quad k \in \mathcal{K}; \\ \sum_{k \in \mathcal{K}} \alpha_k x_i^k = \frac{P_i}{N_0 B}, \quad x_i^k \geq 0, \quad i \in \mathcal{M}, \quad k \in \mathcal{K}. \end{aligned} \quad (6)$$

Given fixed channel gains, denote the optimal sum-rate (normalized by  $B$ ) given by the solution to (6) as  $F^*(\frac{P_1}{N_0 B}, \dots, \frac{P_M}{N_0 B})$ . It is not hard to see that the following holds for  $F^*$ :

**Lemma 2.**  $F^* \geq F$  and  $F^*$  is concave.

Furthermore, any function<sup>4</sup> that is no less than  $F$  and is concave provides an upper bound on  $F^*$ .

**Lemma 3.** For any function  $\tilde{F}$  such that  $\tilde{F} \geq F$  and  $\tilde{F}$  is concave, we have  $\tilde{F} \geq F^*$ .

*Proof:* If we replace  $F$  in the objective of (6) with  $\tilde{F}$ , then the resulting maximum becomes  $B\tilde{F}$ , since  $\tilde{F}$  is concave,

<sup>4</sup>Unless otherwise stated, all functions are defined in the same domain as  $F$ .

which is clearly no less than the original maximum  $BF^*$  from the fact  $\tilde{F} \geq F$ . ■

An upper bound is tight if feasibility also applies.

**Lemma 4.** Suppose that  $\tilde{F} \geq F$  and  $\tilde{F}$  is concave. If  $\tilde{F}$  is feasible at some  $\mathbf{x}$ , i.e., there exist  $\mathbf{x}_i$ 's and  $\alpha_i$ 's with  $\alpha_i \geq 0$  and  $\sum_i \alpha_i = 1$ , such that  $\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$  and  $\tilde{F}(\mathbf{x}) = \sum_i \alpha_i F(\mathbf{x}_i)$ , then  $F^*(\mathbf{x}) = \tilde{F}(\mathbf{x})$ .

*Proof:* Obviously,  $F^*(\mathbf{x}) \geq \tilde{F}(\mathbf{x})$ . On the other hand, by Proposition 3,  $\tilde{F}(\mathbf{x}) \geq F^*(\mathbf{x})$ . Therefore,  $F^*(\mathbf{x}) = \tilde{F}(\mathbf{x})$ . ■ Lemmas 2 and 4 directly imply the following theorem:

**Theorem 1.** A sufficient and necessary condition for a function  $\tilde{F}$  to be optimal, i.e.,  $\tilde{F} = F^*$ , is  $\tilde{F} \geq F$ ,  $\tilde{F}$  is concave, and  $\tilde{F}$  is feasible everywhere.

In other words, to determine the optimal sum-rate, Theorem 1 implies that it is sufficient to search for a concave upperbound to  $F$  which is feasible everywhere. Concavity also implies the following useful properties of  $F^*$ :

**Proposition 1.** If  $F^* = F$  on a convex set, then  $F$  must be concave on this set.

**Proposition 2.** For any finite set of points in the  $M$ -dimensional space,  $\{\mathbf{x}_i\}$ , if there exist some  $\alpha_i$ 's, where  $\alpha_i > 0$  and  $\sum_i \alpha_i = 1$ , such that  $F^*(\sum_i \alpha_i \mathbf{x}_i) = \sum_i \alpha_i F^*(\mathbf{x}_i)$ , then for all  $\beta_i$ 's satisfying  $\beta_i \geq 0$  and  $\sum_i \beta_i = 1$ ,  $F^*(\sum_i \beta_i \mathbf{x}_i) = \sum_i \beta_i F^*(\mathbf{x}_i)$ .

### C. Bounds on the Optimal Sum-Rate

Next, for the two-user case, we discuss lower and upper bounds on  $F^*$ , based on which we determine the optimal spectrum allocation achieving  $F^*$ .

1) *Lower Bound (Optimal FS/FDM):* First we consider the optimal sum-rate when the spectrum allocation is restricted to one of two special cases: FS and FDM. In this restricted setting, the optimal sum-rate can be achieved as indicated in the following Theorem:

**Theorem 2.** For the two-user Gaussian interference channel, when only FS and FDM are considered, the sum-rate maximizing spectrum allocation satisfies the following: (i) If

$$\frac{h_{11}P_1}{N_0B} + \frac{h_{22}P_2}{N_0B} \leq \frac{h_{11}h_{22}}{h_{12}h_{21}} - \frac{h_{11}}{h_{12}} - \frac{h_{22}}{h_{21}},$$

then FS is optimal with both users transmitting with full power.

(ii) Otherwise, FDM is optimal, with user 1 exclusively using a band of  $B\alpha_1$  Hz and user 2 exclusively using the remaining band of  $B\alpha_2 = B(1 - \alpha_1)$  Hz where  $\alpha_1 = h_{11}P_1/(h_{11}P_1 + h_{22}P_2)$ , and both users transmit with full power.

By Theorem 2, the optimal sum-rate (divided by  $B$ ) from FS/FDM, defined as  $F_0(\frac{P_1}{N_0B}, \frac{P_2}{N_0B})$ , is as follows:

$$F_0(x, y) = \begin{cases} F(x, y), \\ \text{if } h_{11}x + h_{22}y < \frac{h_{11}h_{22}}{h_{12}h_{21}} - \frac{h_{11}}{h_{12}} - \frac{h_{22}}{h_{21}}, \\ \log(1 + h_{11}x + h_{22}y), \\ \text{otherwise.} \end{cases}$$

Clearly,  $F_0$  provides a lower bound for  $F^*$ . Note that  $F_0$  cannot be optimal, since it is not concave.

2) *Upper Bound:* Next, we construct an upper bound  $G$  on  $F^*$  from  $F_0$ . For ease of analysis, we normalize the Gaussian interference channel (as in [1], [11]) and define  $a = \frac{h_{21}}{h_{22}}$ ,  $b = \frac{h_{12}}{h_{11}}$ ,  $u = h_{11}x$ , and  $v = h_{22}y$ . Note that the assumption here is  $a + b < 1$ ; otherwise,  $a + b \geq 1$  is the trivial case where FDM as in (ii) of Theorem 2 is always optimal.

Along the line  $u + v = c$  with  $c$  being a constant less than  $1/(ab) - 1/a - 1/b$  (which is positive assuming  $a + b < 1$ ), we can verify that at the point  $(u_0(c), v_0(c))$  where

$$u_0(c) = \frac{c\sqrt{1+ac}}{\sqrt{1+ac} + \sqrt{1+bc}}, \quad v_0(c) = \frac{c\sqrt{1+bc}}{\sqrt{1+ac} + \sqrt{1+bc}}, \quad (7)$$

$F_0$  reaches its maximum, defined as

$$g_0(c) = 2 \log \left( 1 + \frac{c}{1 + \sqrt{(1+ac)(1+bc)}} \right).$$

Along the line  $u + v = c \geq 1/(ab) - 1/a - 1/b$ ,  $F_0$  remains a constant  $g_0(c) := \log(1 + c)$ .

Since  $g_0$  is not concave, we construct a concave function  $g$  such that  $g \geq g_0$ . It is easy to see that there exists unique  $c_1$  and  $c_2$  such that  $0 < c_1 < 1/(ab) - 1/a - 1/b < c_2$  and

$$\frac{g_0(c_2) - g_0(c_1)}{c_2 - c_1} = g'_0(c_1) = g'_0(c_2). \quad (8)$$

Then,  $g$  can be defined as follows:

$$g(c) := \begin{cases} g_0(c_2) + g'_0(c_2)(c - c_2) = \log(1 + c_2) + \frac{c - c_2}{1 + c_2}, & \text{if } c_1 < c < c_2; \\ g_0(c), & \text{otherwise.} \end{cases}$$

Clearly,  $g$  is concave.

Finally, along the line  $u + v = c$ , let  $G$  take the constant value  $g(c)$ , i.e.,  $G(x, y) := g(h_{11}x + h_{22}y)$ .  $G$  is obviously concave, since  $g$  is concave. Also, from the fact that  $G \geq F$  and by Lemma 3,  $G$  provides an upper bound for  $F^*$ .

**Remark:** Note that  $F_0/G$  provides a lower bound on efficiency achievable from the optimal FS/FDM scheme, for which we can verify

$$\frac{F_0}{G} \geq \min_{0 < c \leq \frac{1}{ab} - \frac{1}{a} - \frac{1}{b}} \frac{\log(1 + c)}{2 \log \left( 1 + \frac{c}{1 + \sqrt{(1+ac)(1+bc)}} \right)}.$$

The right-hand side is strictly increasing in  $a$  and  $b$ , and is always larger than  $1/2$ . Therefore, the optimal FS/FDM scheme achieves at least half of the optimal sum-rate, and its efficiency approaches 1 as  $a + b \rightarrow 1$ .

#### D. Optimal Spectrum Allocation

Finally, we can determine the optimal spectrum allocation from  $G$ . Fig. 2 shows the triangle with the three corner points being  $(c_2, 0)$ ,  $(0, c_2)$ , and  $(u_0(c_1), v_0(c_1))$ , labeled as  $A$ ,  $B$ , and  $C$ , respectively, for an example with  $a = 0.1$  and  $b = 0.3$ . Note that  $c_1$ ,  $c_2$ , and point  $C$ ,  $(u_0(c_1), v_0(c_1))$ , are calculated from (7) and (8). It is worth mentioning that this triangle is

exactly the triangle representing scenario 3 as shown in Fig. 1. The curve from the origin to point  $C$  depicts  $(u_0(c), v_0(c))$  as  $c$  varies from 0 to  $c_1$ , which again is computed from (7).

Recall that for  $h_{11}x + h_{22}y \geq c_2$ ,  $G(x, y) = F_0(x, y) = \log(1 + h_{11}x + h_{22}y)$ . This means that in the area above the triangle, which corresponds to scenario 1 in Fig. 1,  $G$  can be achieved using FDM. Thus, by Lemma 4,  $F^* = G$  and so FDM is optimal in this region, which is summarized as follows:

**Theorem 3.** *If  $\frac{h_{11}P_1}{N_0B} + \frac{h_{22}P_2}{N_0B} \geq c_2$ , where  $c_2$  is obtained in (8), then the optimal sum-rate is  $B \log(1 + \frac{h_{11}P_1 + h_{22}P_2}{N_0B})$ , which is achieved using FDM as in (ii) of Theorem 2.*

Next, since  $g$  is linear from  $c_1$  to  $c_2$ ,  $G$  is linear over the triangle-region, corresponding to scenario 3 in Fig. 2. Furthermore, at the three corner points  $A$ ,  $B$ , and  $C$ ,  $G = F$ . Therefore,  $G$  is feasible in this triangle, and is achievable from a convex combination of  $F$  at the three corner points. Again, by Lemma 4,  $F^* = G$  in this region.

**Theorem 4.** *If  $(\frac{h_{11}P_1}{N_0B}, \frac{h_{22}P_2}{N_0B}) = \alpha_1(c_2, 0) + \alpha_2(0, c_2) + \alpha_3(u_0(c_1), v_0(c_1))$ , where  $\alpha_i > 0$ ,  $i = 1, 2, 3$ ,  $\sum_{i=1}^3 \alpha_i = 1$ , and  $c_1$ ,  $c_2$ ,  $u_0$ , and  $v_0$  are determined by (7) and (8), then the optimal sum-rate is*

$$B\alpha_1 \log(1 + c_2) + B\alpha_2 \log(1 + c_2) + B\alpha_3 \left[ \log \left( 1 + \frac{u_0(c_1)}{1 + av_0(c_1)} \right) + \log \left( 1 + \frac{v_0(c_1)}{1 + bu_0(c_1)} \right) \right],$$

which can be achieved by allocating  $B\alpha_1$  exclusively to user 1 and  $B\alpha_2$  exclusively to user 2, while letting them share  $B\alpha_3$  at the corresponding power levels.

Now for the area below the triangle, note that along the curve from the origin to point  $C$  as shown in Fig. 2,  $G = F$ , which is feasible by FS. By Lemma 4,  $F^* = G$  and FS is optimal along this curve. Along the two lower edges of the triangle, i.e., from  $C$  to  $A$  and  $C$  to  $B$ ,  $G$  can be achieved from a convex combination of  $F$  at the two end points of the corresponding edge, and therefore  $F^* = G$  by letting two users share one sub-band and allocating the left band for one user's exclusive use. Finally, combining the preceding facts with Propositions 1 and 2, we argue that for scenario 2 in Fig. 2, FS at the full powers is optimal, while for scenarios 4 and 5, it is optimal to have the users share one sub-band and the higher power user to use the remaining band exclusively. However, the boundary of scenario 2 and the optimal spectrum allocations for power constraints falling in scenario 4 or 5 need to be determined numerically, e.g., through a bisection search, which is omitted due to space considerations.

#### E. Remark on Multiple Users

We briefly discuss the multiple-user case. Recall that the analysis in III-B applies to multiple users. That is, the power constraints must be tight at the optimal solution, and the same criteria for the optimal sum-rate holds. However, further extension to multiple users of the preceding method seems

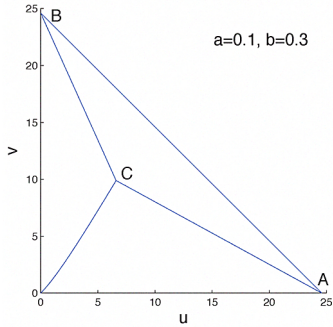


Fig. 2. Regions illustrating the optimal spectrum allocation for a standard two-user channel with cross-channel gains 0.1 and 0.3.

difficult. Alternatively, the algorithm presented next can be applied for numerical computation of the optimal spectrum allocations for multiple users.

#### IV. NUMERICAL ALGORITHM

In this section, we present an *alternating bandwidth and power update* (ABP) algorithm as follows:

- 1) Initialize bandwidth and power allocation.
- 2) Fix the bandwidth allocation. Users update powers simultaneously using the gradient projection method.
- 3) Fix the powers. Users jointly update the bandwidth allocation using gradient projection.
- 4) Go to 2 and repeat, until the algorithm converges.

It can be proved that the ABP algorithm converges to a local optimum for small enough power and bandwidth step sizes. Details of the algorithm and the proof of convergence will be presented in the extended version of this paper.

Simulation results for two users always indicate convergence to the global optimum with appropriately chosen step sizes. For three users, Fig. 3 shows convergence curves and the resulting spectrum allocations for an example with  $N_0B = 1$ ,  $P_1 = P_2 = P_3 = 10$ , and different channel gains (the channel gain matrix  $h$  is as shown in the figure). We point out that for the bottom case shown in Fig. 3, the result cannot be optimal with user 3 using no power instead of full power.

#### V. CONCLUSION

In this work, we completely characterized the optimal spectrum allocation maximizing the sum-rate for a two-user flat-fading Gaussian interference channel, treating interference as noise. An alternating bandwidth and power update algorithm was proposed for multiple users, with guaranteed convergence to a local optimum.

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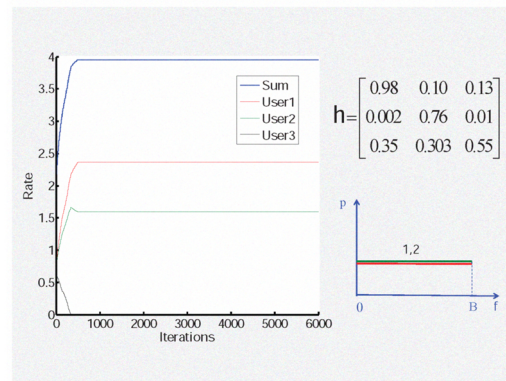
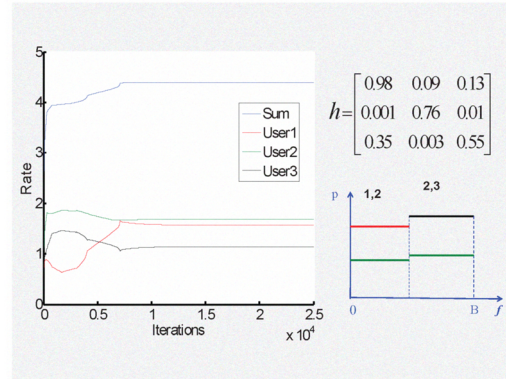


Fig. 3. Convergence results of the ABP algorithm for three users with different channel gains (indicated by the  $h$  matrix). The resulting spectrum allocation is also shown.

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