**Northwestern University**
Department of Electrical and Computer Engineering

ECE 510 Spring 2005

**Problem set 4:** Due 5/17/05

**Problem 1:** In this problem we again consider the problem of bandwidth scaling in a multi-path fading channel. As in lecture, consider dividing a multi-path fading channel into a set of $L$ parallel channels, each in a different coherence band. Then we consider a discrete-time model and assume that each time sample corresponds to a different coherence-time. For a given coherence-time, let $\mathbf{Y} = (y_1, \dots, y_L)^T$ be the complex vector of received signals, where $y_l$ represent the received signal in the $l$th coherence band. Likewise let $\mathbf{X}$ be $L$ dimensional vector representing the channel input in each coherence band. These are related by

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W},$$

where $\mathbf{H}$ is a $L \times L$ diagonal matrix with the $l$th diagonal entry $h_l$ corresponding to the channel gain in the $l$th coherence band, and $\mathbf{W} = (w_1, \dots w_L)^T$ represents the additive noise. Assume that $\{h_l\}$ are i.i.d. complex Gaussians, with distribution $CN(0,1)$ and $\{w_l\}_l$ are also i.i.d. complex Gaussians with distribution $CN(0, N_o)$. Finally, assume that the input has a power constraint given by

$$\sum_{l=1}^{L} \mathrm{E}(|x_l|^2) \leq P.$$

The capacity of this channel (assuming no CSI at the receiver or transmitter) is given by the maximum of $I(\mathbf{X};\mathbf{Y})$, over all probability distributions on the input $\mathbf{X}$ that satisfy the power constraint.

a. Using that $I(\mathbf{H},\mathbf{X};\mathbf{Y}) = I(\mathbf{H};\mathbf{Y}|\mathbf{X}) + I(\mathbf{X};\mathbf{Y})$, show that

$$I(\mathbf{X};\mathbf{Y}) \leq L \log\left(1 + \frac{P}{LN_o}\right) - \sum_{l=1}^{L} \mathrm{E} \log\left(1 + \frac{|x_1|^2}{N_0}\right).$$

b. Assume the transmitter must allocate equal power to each coherence band, i.e., $\mathrm{E}(|x_l|^2) = P/L$ for all $l = 1, \dots, L$. Using the bound in part (a), show that the capacity of the channel (with this power allocation) must go to zero as $L \to \infty$.

c. Now assume that the transmitter uses "peaky" signalling and transmitts with power $\lambda$ in only one of the frequence bands. In this case show that:

$$\lim_{L\to\infty} I(\mathbf{X};\mathbf{Y}) \geq \frac{\lambda}{N_o} - \log\left(1 + \frac{\lambda}{N_o}\right).$$

d. As in lecture, now assume that the transmitter only transmitts a fraction of the time, given by $\theta$. Show that as $\theta \to 0$, the mutual information converges to $\frac{P}{N_0}$ nats/channel use.

**Problem 2:** This problem reviews some facts about complex random vectors. Let $\mathbf{X}$ be a $n$-dimensional complex random vector, i.e., a vector where each entry is a complex random variable, with a given joint distribution. $\mathbf{X}$ is *isotropically distributed* if $U\mathbf{X}$ has the same probabilitiy density as $\mathbf{X}$ for any (deterministic) unitary matrix $U \in \mathbb{C}^{n\times n}$ (Recall a square matrix is unitary if $U^*U = I$, where $U^*$ denotes the congugate transpose). $\mathbf{X}$ is Gaussian if its components are jointly Gaussian random variables. $\mathbf{X}$ is *proper* if its pseudo covariance matrix $\mathrm{E}\mathbf{X}\mathbf{X}^T$ is the all zero matrix. The distribution of a proper, complex Gaussian random vector with covariance matrix $Q$ and mean $\mathbf{m}$ is denoted $CN(\mathbf{m}, Q)$. If $\mathbf{X}$ has distribution $CN(\mathbf{m}, Q)$ with $Q$ non-singular, then its pdf is given by

$$f_{\mathbf{X}}(X) = \det(\pi Q)^{-1} \exp(-(X - \mathbf{m})^* Q^{-1}(X - \mathbf{m})).$$

a. Suppose that each entry of $\mathbf{X}$ is i.i.d. with distribution $CN(0, \sigma^2)$. Show that $\mathbf{X}$ is isotropically distributed.

b. Suppose that $\mathbf{X}$ has distribution $CN(\mathbf{0}, Q)$ and is isotropically distributed. Show that the entries of $\mathbf{X}$ are i.i.d. with distribution $CN(0, \sigma^2)$. (recall a covariance matrix is always nonegative definite and Hermitian.)