Poisson processes, Markov chains and M/M/1 queues

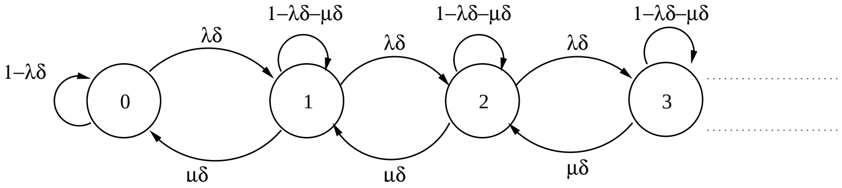
Advanced Communication Networks

Lecture 5

M/M/1 Analysis

Poisson Arrivals, Exponential service times, 1 server (FIFO)

- Look at discrete times δ
- With high prob., only one arrival or departure
- *Discrete-time Markov chain*



Analysis Contd..

Steady state prob.s $\{p_n\}$

Balance equations

$$p_n \mu \delta = p_{n-1} \lambda \delta$$

$$\Rightarrow p_n = \frac{\lambda}{\mu} p_{n-1} = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0$$

$$\sum_n p_n = 1 \ \Rightarrow p_0 = 1 - \rho$$

$$\boxed{p_n = (1-\rho)\rho^n}$$

Stability of system: $\lambda < \mu$

Computing system averages

Comment: $p_n \neq f(\delta)$
As $\delta \searrow 0$ same answers for cont. time model

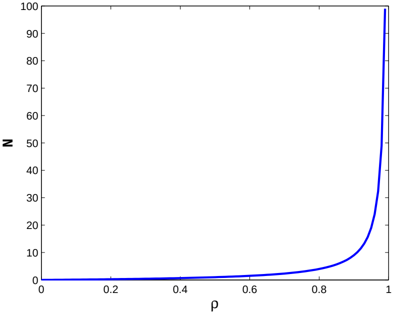
Average number in system

$$N = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho)$$
$$= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$$

Average system delay: Little's law

$$T = \frac{N}{\lambda} = \frac{1}{\mu - \lambda}$$

As $\rho \to 1$, $N, T \to \infty$



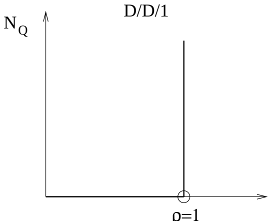
- Single pole response - typical of queuing systems

Other system variables

$$W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} \qquad N_Q = \lambda W = \frac{\rho^2}{1-\rho}$$

Compare to D/D/1 queue

one arrival every $\frac{1}{\lambda}$ sec, 1 departure every $\frac{1}{\mu}$ sec



- $N_Q = 0 \quad \rho < 1.$
- M/M/1: queue size, delay blows up for ρ near 1

Intuition: Variability causes performance loss

Changing transmission rate

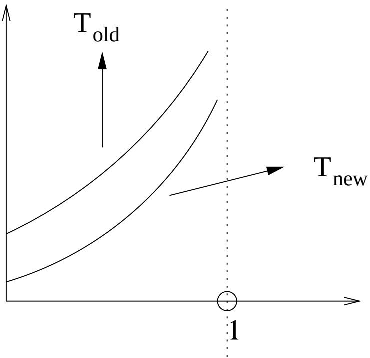
M/M/1 queue: Arrival rate and Service time is doubled

What happens to delay? N?

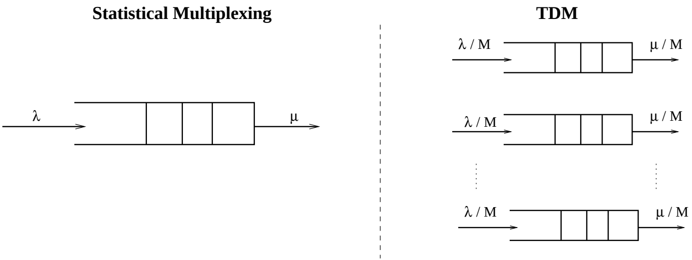
$$\lambda \rightarrow 2\lambda, \quad \mu \rightarrow 2\mu \quad \Rightarrow \rho \text{ stays same}$$

$$N = \frac{\rho}{1-\rho} \quad \Rightarrow N \text{ stays same}$$

$$T = \frac{1}{\mu - \lambda} = \frac{\frac{1}{\mu}}{1-\rho} \quad \Rightarrow T \text{ reduces by } \frac{1}{2}$$


T_{old}
T_{new}
1

Example 2



M independent Poisson streams, rate $\frac{\lambda}{M}$

$$T_{SM} = \frac{1}{\mu - \lambda} \quad T_{TDM} = \frac{m}{\mu - \lambda}$$

Delay reduced by factor of $m \Rightarrow$ “Statistical Multiplexing gain”

Cons: Difficult to isolate *Bad* flows ; Provide guarantees

Distribution of System Variables

What about distribution for N?
For eg: Variance

$$Var(N) = \sum_{n=0}^{\infty} \rho^n (1-\rho) \cdot n^2 - \left(\frac{\rho}{1-\rho}\right)^2$$
$$= \frac{\rho}{(1-\rho)^2}$$

Likewise, Distn. for T in Prob. 3.1 (M/M/1 queues)

PASTA property

Interested in state of system *just before packet arrives*

Eg: to calculate Blocking probability

Steady-state prob. arriving packet sees in the system

$$\lim_{t \to \infty} \Pr(N(t) = n | \text{arrival } @t^{+}) \ ?$$

Is'nt this same as p_n?

Not necessary

Eg: D/D/1 system, $\lambda < \mu$
Above prob. is zero, but no steady-state exists

PASTA property Contd..

For Poisson traffic,
The two quantities are equal $\Rightarrow$ “PASTA”

Proof

$$
\begin{aligned}
a_n &= \mathrm{P}\{N(t) = n | \text{arrival } @t^+\} \\
&= \mathrm{P}\{N(t) = n | A(t, t+\delta)\} \\
&= \frac{\mathrm{P}(N(t)=n, A(t,t+\delta))}{\mathrm{P}(A(t,t+\delta))} = \frac{\mathrm{P}\left(A(t,t+\delta) | N(t) = n\right) \mathrm{P}\left(N(t) = n\right)}{\mathrm{P}\left(A(t,t+\delta)\right)}
\end{aligned}
$$

But $A(t, t+\delta)$ ind. of $N(t) = n \Rightarrow \; a_n(t) = \mathrm{P}\left(N(t) = n\right)$

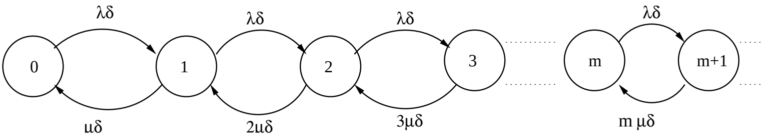
Holds for broad class of queuing systems w/ Poisson arrivals, ind service distribution (Memoryless property)

M/M/1 - Last slide

What is the prob arriving customer finds system empty?

$p_0 = 1 - \rho$

Other Markovian systems: M/M/m



m servers in system

Given 2 packets in service,
Prob of departure = Prob(1st packet departs) + Prob(2nd packet departs)
= $\mu\delta + \mu\delta$

Service time of state $n = m\mu \quad n > m$

Ex: Circuit switched networks, blocked calls wait

M/M/m analysis

Balance equations

$$
\begin{aligned}
p_{n-1}(\lambda\delta) &= p_n(n\mu\delta) \quad n = 1, 2, \cdots, m \\
p_{n-1}(\lambda\delta) &= p_n(m\mu\delta) \quad n > m \\
\Rightarrow p_n &= \left(\frac{\lambda}{n\mu}\frac{\lambda}{(n-1)\mu}\cdots\frac{\lambda}{\mu}\right) p_0 \quad n \leq m \\
p_n &= \left(\frac{\lambda}{m\mu}\right)^{n-m}\frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m p_0 \quad n > m
\end{aligned}
$$

M/M/m analysis Contd..

$$p_n = \begin{cases} \frac{1}{n!}(m\rho)^n p_0 & n \le m \\ \frac{m^m \rho^n}{m!} & n > m, \end{cases}$$

If stable: $\sum_{n=0}^{\infty} p_n = 1$

Stability condition: $\rho = \frac{\lambda}{m\mu} < 1$

$$p_0 = \left(\sum_{n=0}^{m-1} \frac{(m\rho)^n}{n!} + \frac{(m\rho)^m}{m!(1-\rho)} \right)^{-1}$$

Erlang C formula

What is the prob arriving packet has to wait for service?
Same as prob that servers are busy (Follows from PASTA property)

$$P_Q = \Pr(\text{Queuing}) = \Pr(N \geq m)$$
$$= \sum_{n=m}^{\infty} p_n = p_0 \frac{(m\rho)^m}{m!(1-\rho)}$$

- Widely used in telephony
- Model for Blocked delay calls
- Formula also hold for M/G/1 systems (Invariance property)