Solution outline to Homework 7

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1 Problem 7.3

There are three materials $1, 2, 3$. The volume variables associated with them are x_1, x_2, x_3. Constraints are as follows

$$\begin{aligned} 2x_1 + 1x_2 + 3x_3 &\leq 100 \\ x_1 + x_2 + x_3 &\leq 60 \\ x_1 &\leq 40 \\ x_2 &\leq 30 \\ x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The cost function is given as

$$\text{Maximize } 1000x_1 + 1200x_2 + 12000x_3$$

2 Problem 7.8

I had already sent portions of the formulation in my email. Points denote the set of all points where each point is a 2-tuple (x_i, y_i). Constraints of the linear program are as follows

$$\begin{aligned} &\forall i \in Points \\ &\quad e_i \geq ax_i + by_i - c \\ &\quad e_i \geq -(ax_i + by_i - c) \\ &\forall i \in Points \\ &\quad M \geq e_i \end{aligned}$$

The cost function is given by

$$\text{Minimize } M$$

3 Problem 7.11

Dual LP

$$\begin{aligned} &\text{Minimize } 3\lambda + 5\nu \\ &\text{Subject to} \\ &2\lambda + \nu \geq 1 \\ &\lambda + 3\nu \geq 1 \\ &\lambda, \nu \geq 0 \end{aligned}$$

Solution of primal LP (Solve graphically) is $x = \frac{4}{5}$, $y = \frac{7}{5}$. Similarly solution of dual LP is $\lambda = \frac{2}{5}$, $\nu = \frac{1}{5}$. By LP duality verify that at these points the cost functions of primal and dual LP computes to the same value of 2.2.

4 Problem 7.17

(a)Max flow : 4+2+2+3 = 11, Min Cut : {s,a,b}, {c,d,t}
(b)Vertices reachable from S are {A,B} while vertices from T are {C}
(c)Bottleneck Edge Identification: {AC,BC}
(d)The example is shown in Figure 1
(e)Algorithm outline for Bottleneck Edge Idenitification: Generate the residual graph after the ford-fulkerson

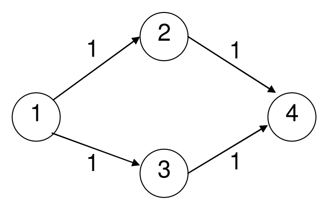

Figure 1: Counterexample for Shortest Path Tree

algorithm is run. Consider each edge $(source, sink)$ in the residual graph where $source, sink \in V$. If s is reachable from $source$ and t is reachable from $sink$ then $(source, sink)$ is a bottleneck edge. Reachability property proves the correctness of the algorithm since if s and t are not reachable from $source$ and $sink$ it is not possible to augment more flow on the path from s to t and then increasing the capacity of edge $(source, sink)$ doesn't increase the maximum flow.

5 Problem 7.25

(a)We define flow variables f_{ij} on each edge $(i, j) \in E$ of the graph G=(V,E). Based on the that maximum flow from S to T can be written as a linear program as follows

$$
\begin{aligned}
\text{Maximize } & f_{sa} + f_{sb} \\
\text{Subject to} & \\
f_{sa} &\leq 1 \\
f_{sb} &\leq 3 \\
f_{ab} &\leq 1 \\
f_{at} &\leq 2 \\
f_{bt} &\leq 1 \\
f_{sa} &= f_{ab} + f_{at} \\
f_{sb} + f_{ab} &= f_{bt} \\
f_{sa}, f_{sb}, f_{ab}, f_{at}, f_{bt} &\geq 0
\end{aligned}
$$

(b)Dual of the linear program: We have dual variables λ_{ij} associated with each edge $(i,j) \in E$ and ν_i associated with each vertex $i \in V$ in the graph G=(V,E).

$$\begin{aligned}
\text{Minimize } \lambda_{sa} + 3\lambda_{sb} + \lambda_{ab} + 2\lambda_{at} + \lambda_{bt} \\
\text{Subject to} \\
\lambda_{sa} + \nu_a \geq 1 \\
\lambda_{sb} + \nu_b \geq 1 \\
\lambda_{ab} - \nu_a + \nu_b \geq 0 \\
\lambda_{at} - \nu_a \geq 0 \\
\lambda_{bt} - \nu_b \geq 0 \\
\lambda_{sa}, \lambda_{sb}, \lambda_{ab}, \lambda_{at}, \lambda_{bt} \geq 0 \\
\nu_a, \nu_b \text{ Unrestricted}
\end{aligned}$$

(c)General dual formulation can be written as

$$\text{Minimize} \sum_{\forall e \in E} c_e y_e \tag{1}$$
$$\text{Subject to} \tag{2}$$
$$y_{ij} - x_i + x_j \geq 0 \forall (i,j) \in E, i,j \notin \{s,t\} \tag{3}$$
$$y_{si} + x_i \geq 1 \forall (s,i) \in E \tag{4}$$
$$y_{jt} - x_j \geq 0 \forall (j,t) \in E \tag{5}$$
$$y_{ij} \geq 0 \forall (i,j) \in E \tag{6}$$
$$\tag{7}$$

(d)Summing up constraints 1-3 we can obtain that

$$\sum_{\forall (i,j) \in E} y_{ij} \geq 1$$

(e)x_u is the potential defined on the vertices. y_e represents if the specified edge is on a cut. Given a graph G=(V,E), for any cut $V = V1 \cup V2$. For an edge (u,v) if $u \in V1$ and $v \in V2$ then $y_{(u,v)} = 1$ othewise $y_{(u,v)}$ is 0. For all nodes $u \in V1$ set x_u as 1 and for all nodes in $u \in V2$ set x_u as 0. The assignment satisfies the constraints given and the cost function $\sum_{\forall e \in E} c_e y_e$ exactly sums up the capacities of edges in a cut.

6 Problem 7.31

(a)2000 augmenting paths can be found out each sending a single unit of flow in this particular graph.
(b)We define the capacity of a s-t path as the smallest capacity of its constituent edges. We define a variable *cap* on each vertex.

```
procedure IdentifyFattestPath(G,c_e,s)
Input:  Graph G=(V,E), c_e capacity of edge e
    source vertex s
Output:  Fattest s-t path
for all vertices v ∈ V:
  cap(v) = ∞
  prev(v) = nil
H = makequeue(V) (using cap values as keys)
while H is not empty:
```

```
u = deletemax(H)
for all edges (u,v) ∈ E
  if cap(v) < min{cap(u), c_(u,v)}
     cap(v) = min{cap(u), c_(u,v)}
     prev(v) = u
     decresekey(H,v)
```

(c)Using the duality of Maximum flow problem we know that there is a min-cut associated with the maximum flow. Now min-cut divides the vertices V into two sets $V1$ and $V2$. Each edge e in the cut must be an edge of some $s-t$ path in the graph and the flow associated with this path is capacity of the edge c_e. Therefore the maximum flow must be the sum of the capacity of edges corresponding to min-cut edges e. Since edges in a min-cut are bounded by $|E|$, therefore maximum flow can be decomposed into at most $|E|$ paths.
(d)Since F is the sum of the flow over at most $|E|$ paths, therefore each flow augmentation decreases the maximum flow bound by $\frac{1}{|E|}$. After k iteration the maximum flow bound is given by

$$F(1-\frac{1}{|E|})^k$$

Following discussion in greedy set cover algorithm, k is given by $\mathrm{O}(|E|\log F)$.